CS103
Winter 2025

Lecture 13:

Mathematical Induction

Part 2 of 2

Outline for Today

- ***“Build Up” versus “Build Down”***
  - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
  - When one assumption isn’t enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

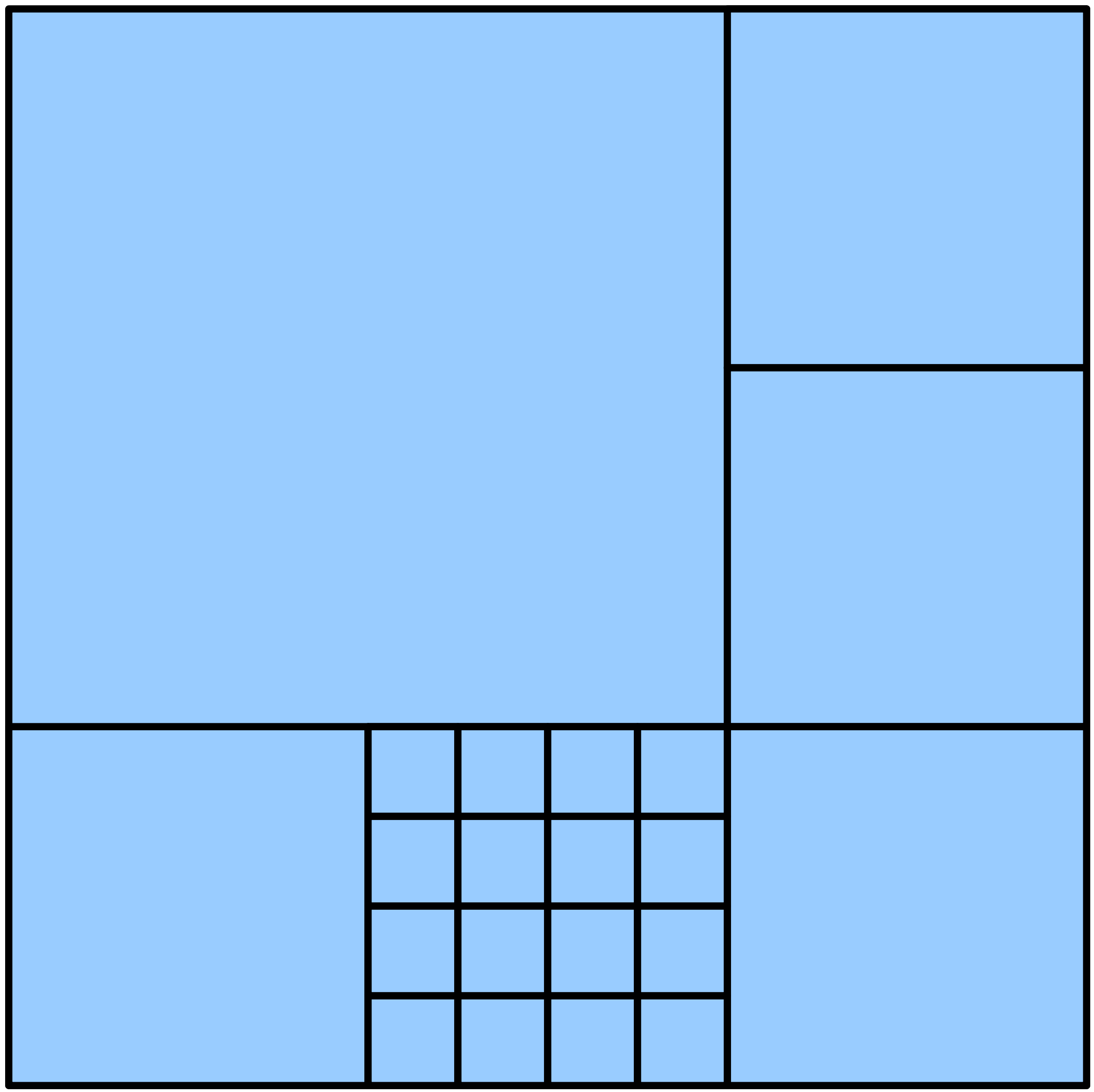
then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Variations on Induction

Subdividing a Square



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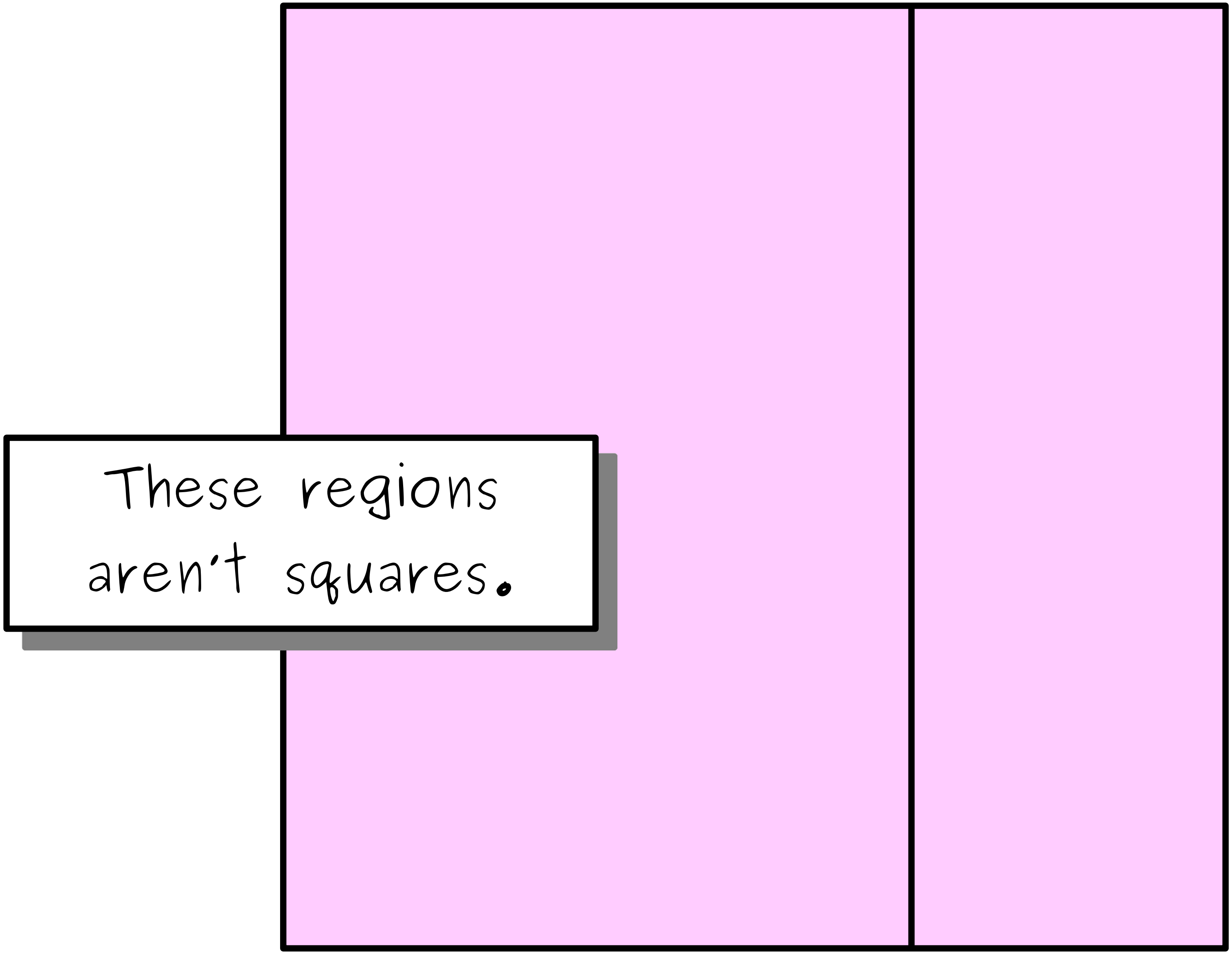

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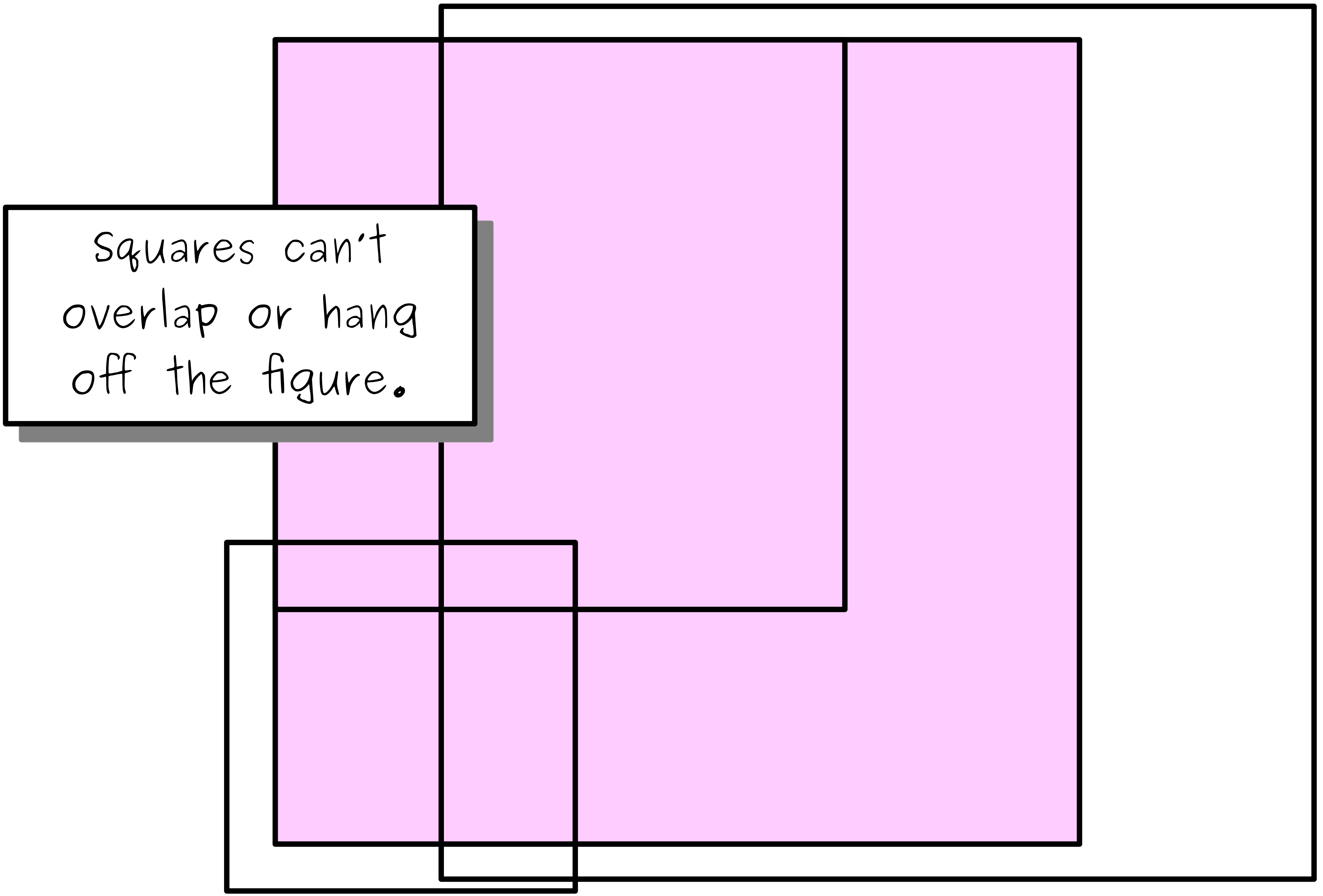
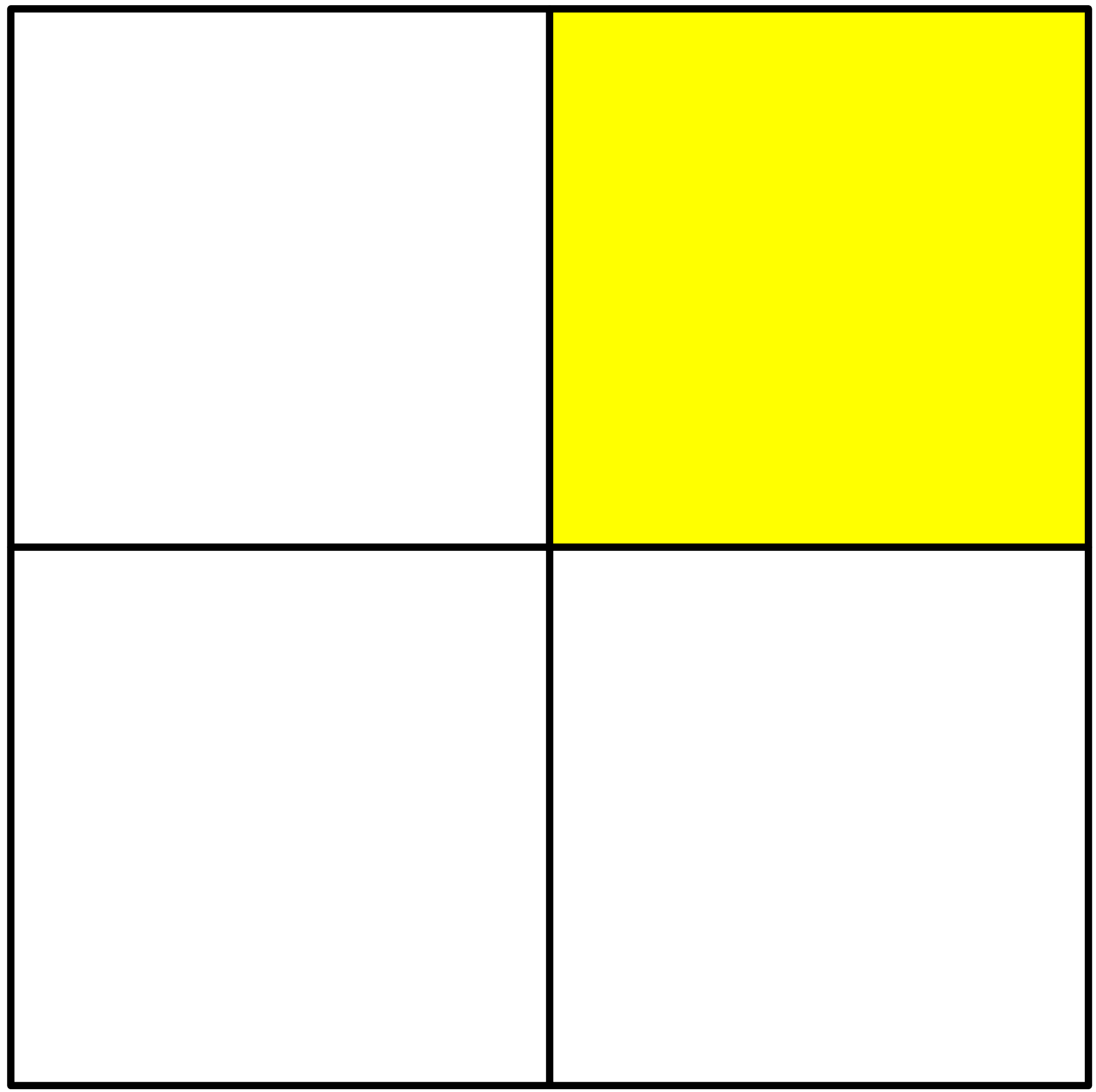


For what values of n can a square be subdivided into n squares?

An Insight



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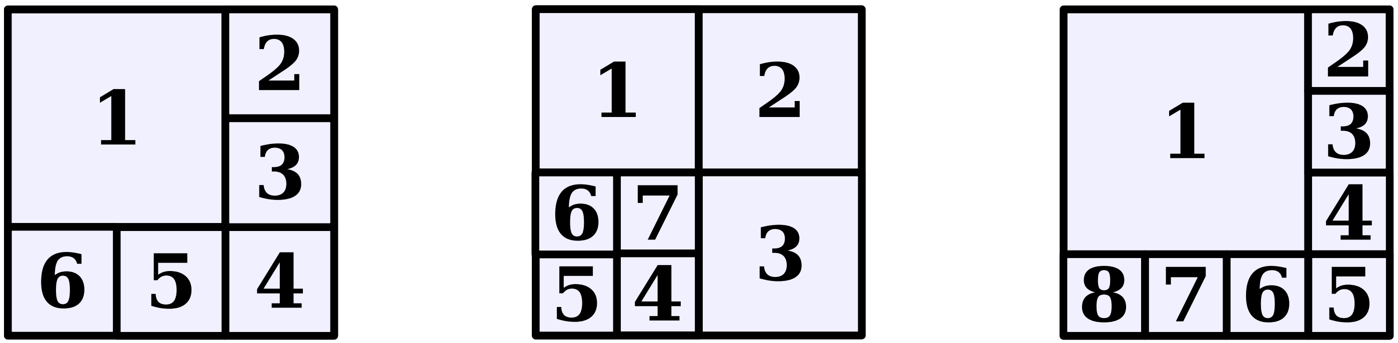

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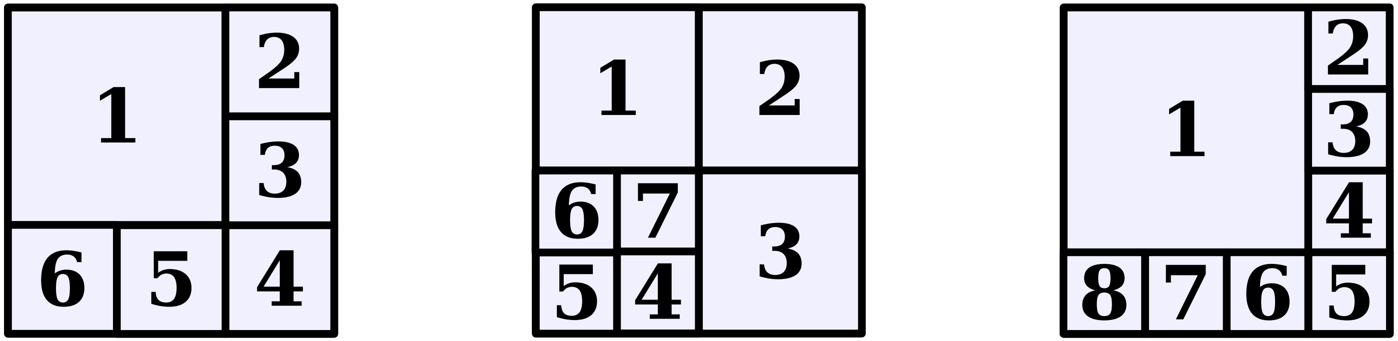

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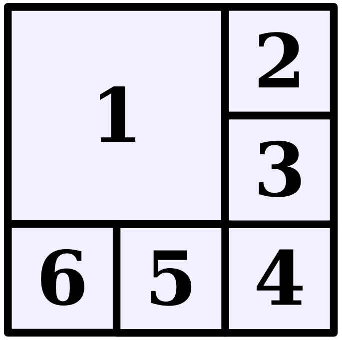

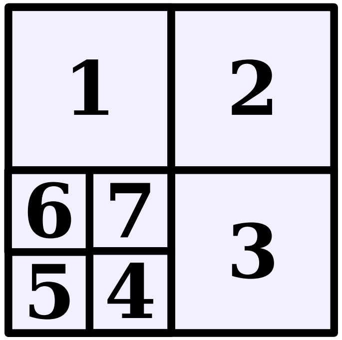

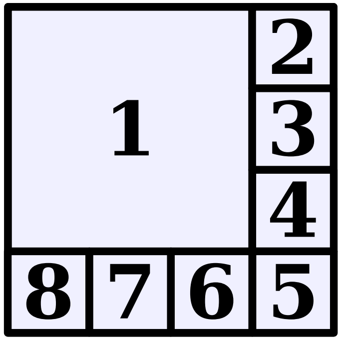

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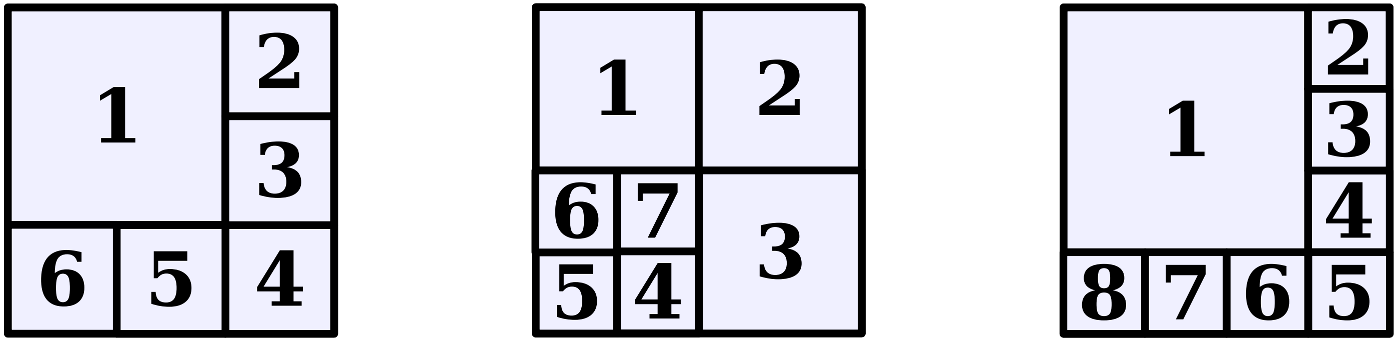

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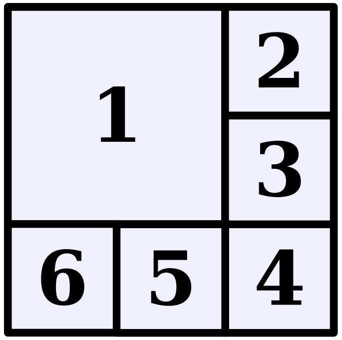

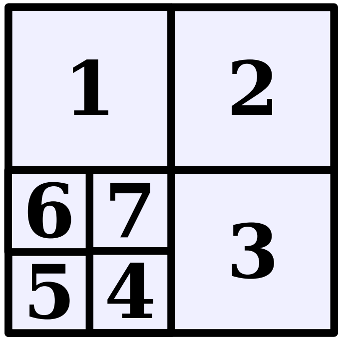

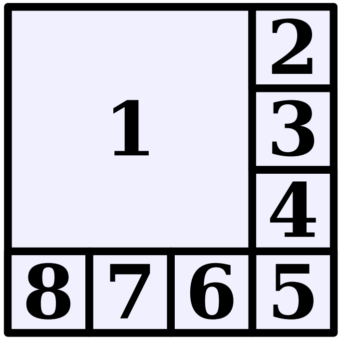

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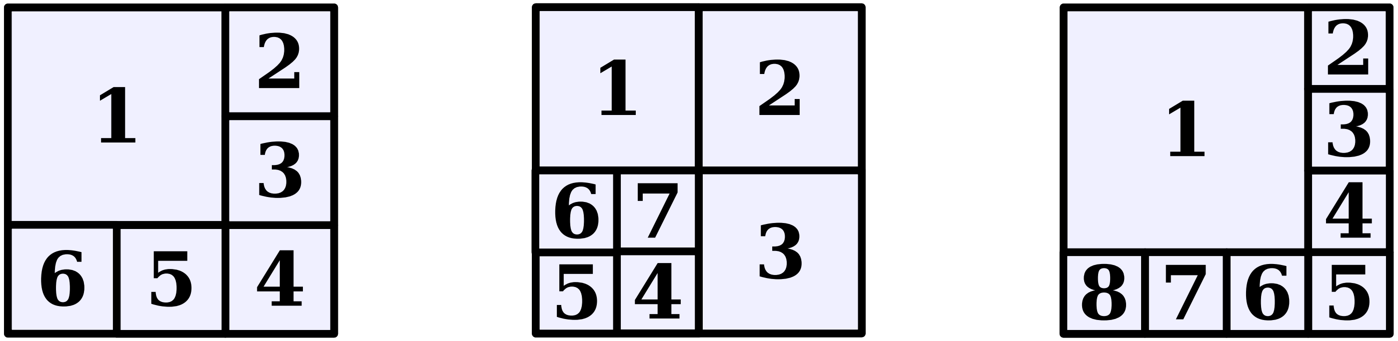

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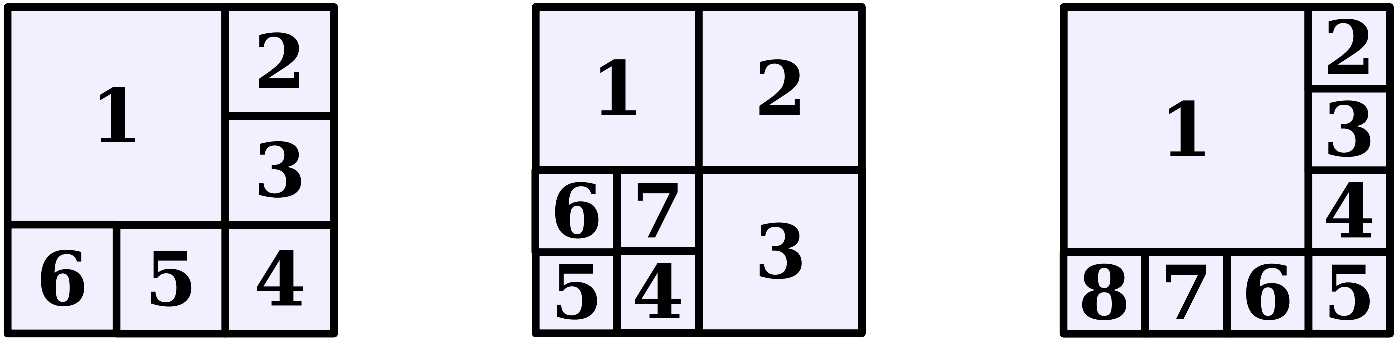

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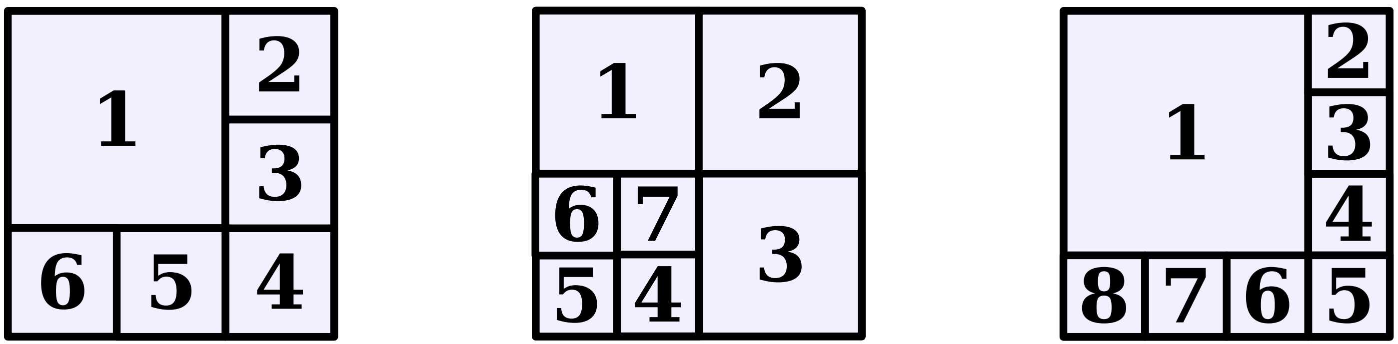

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Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

Quick Announcements!

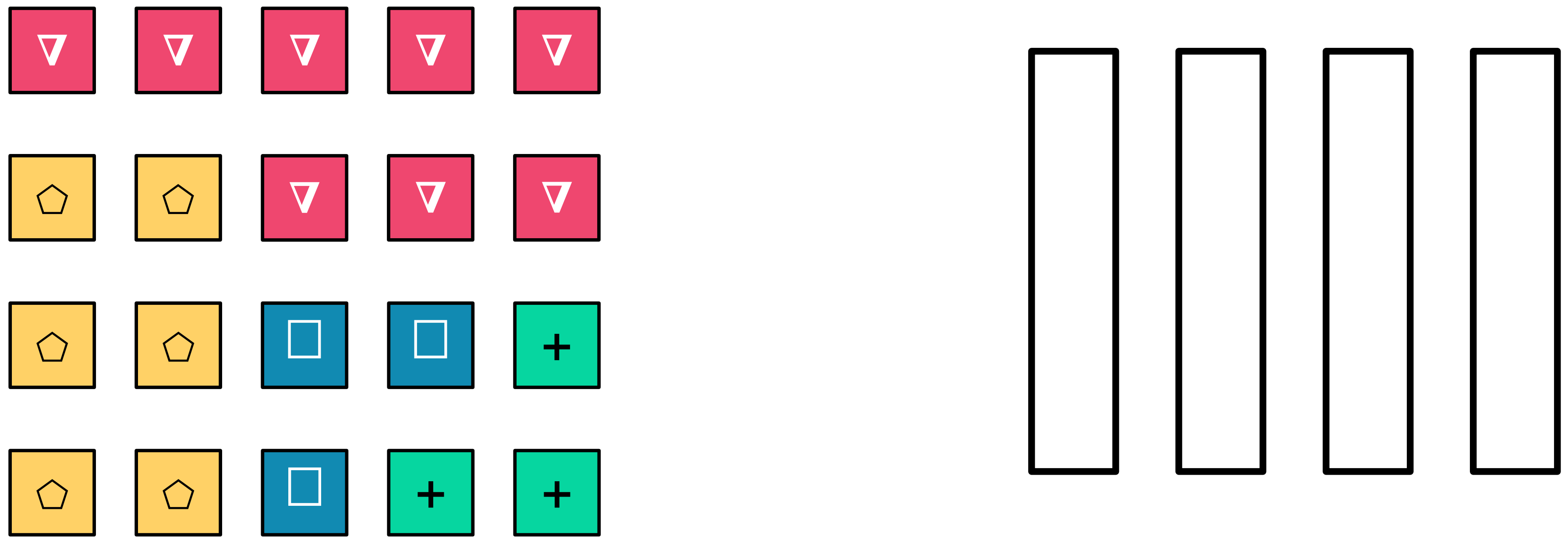
Midterm 1

- You’re done with the midterm! Congrats!
- We will be grading the exam over the weekend and will release solutions and statistics as soon as they’re ready.
- We’re happy to discuss the problems in office hours.

Problem Set Five

- PS5 will be posted today and is due at the normal Friday 1:00PM time next week.
  - You can use a late day to extend the PS4 deadline to Saturday at 1:00PM if you'd like.
- You know the drill: ask questions on Ed or office hours if you have them. That's what we're here for!

The Colored Cubes Problem



Here are 20 cubes of 4 different colors.
Split them into 4 groups of 5 cubes each so that
each group has cubes of at most two different colors.

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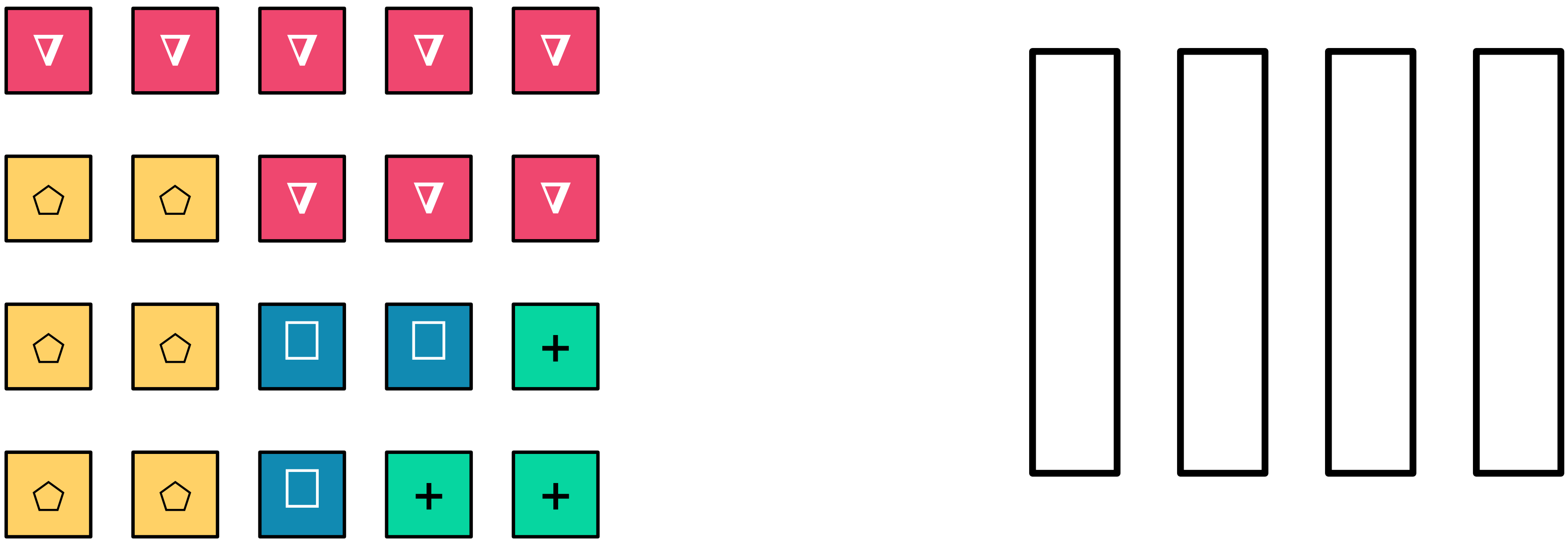
The Magic Potions Problem



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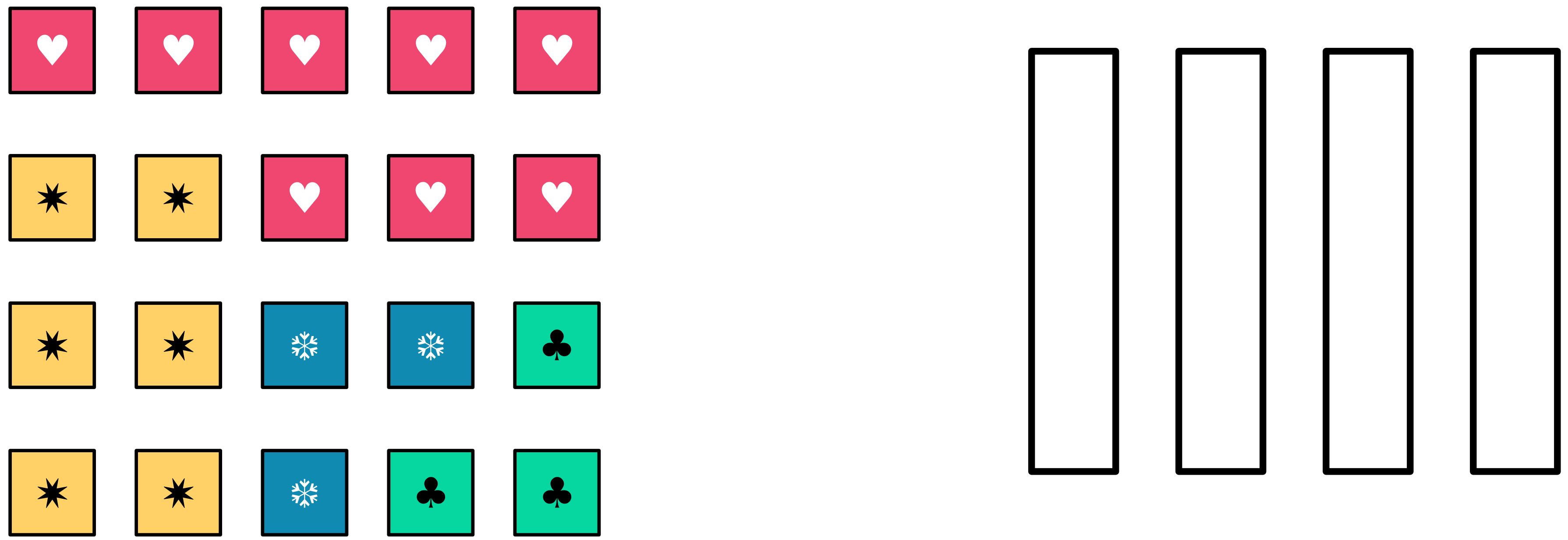

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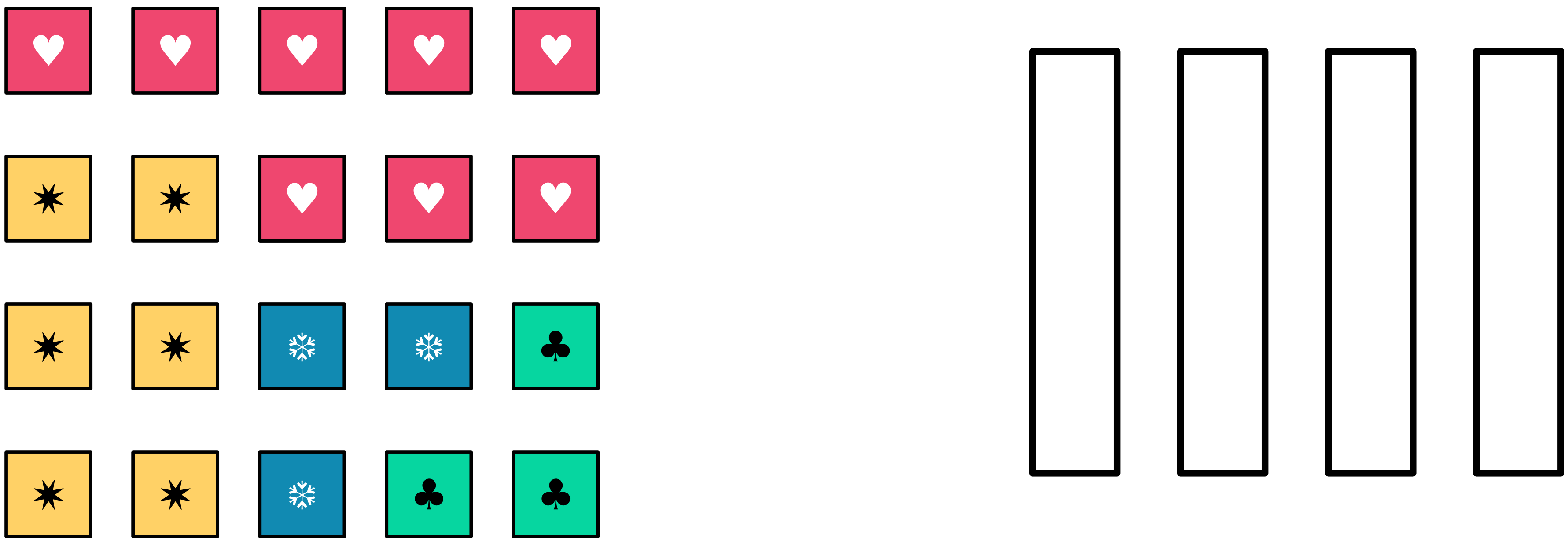

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Split them into 4 **shelves** of 5 **potions** each so that each **shelf** has **potions** of at most two different **colors**.

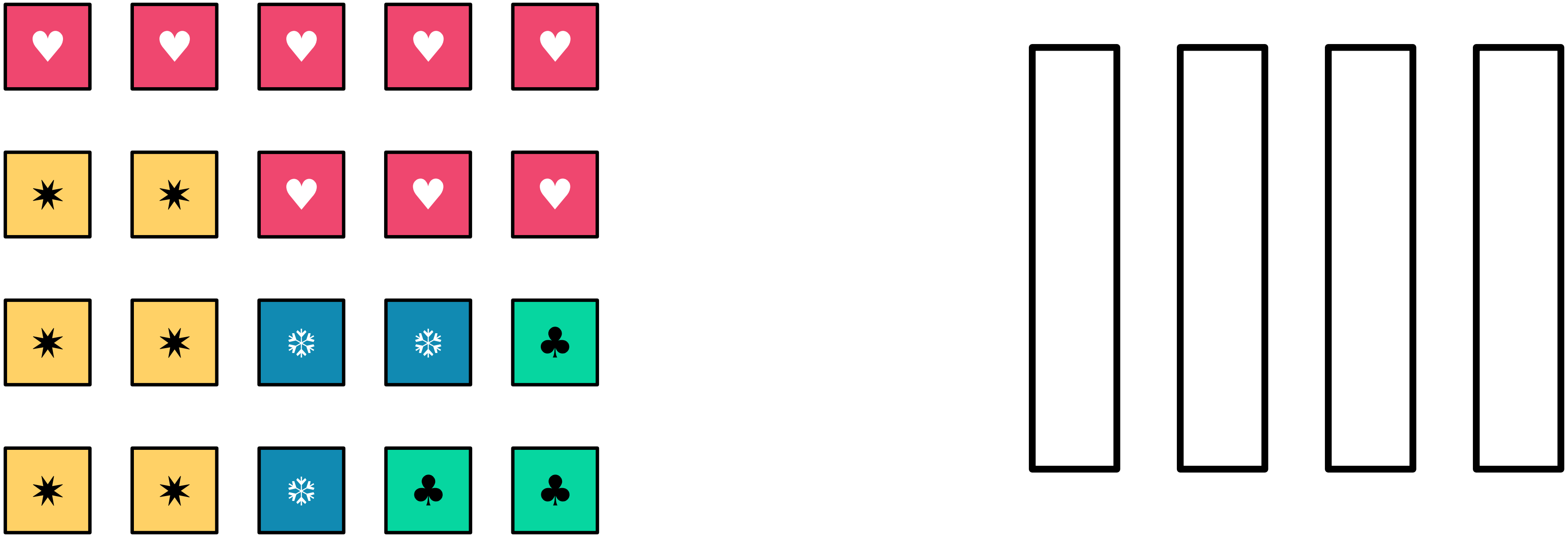
To be clear, here's the setup:

Each shelf will ***always*** have 5 potions.

A shelf will ***never*** have potions of
more than 2 different colors.

We'll always have **$5n$ potions** of **n colors**
(where $n \in \mathbb{N}$).

Currently, we have **20 potions** and **4 colors**
(so, $n = 4$).



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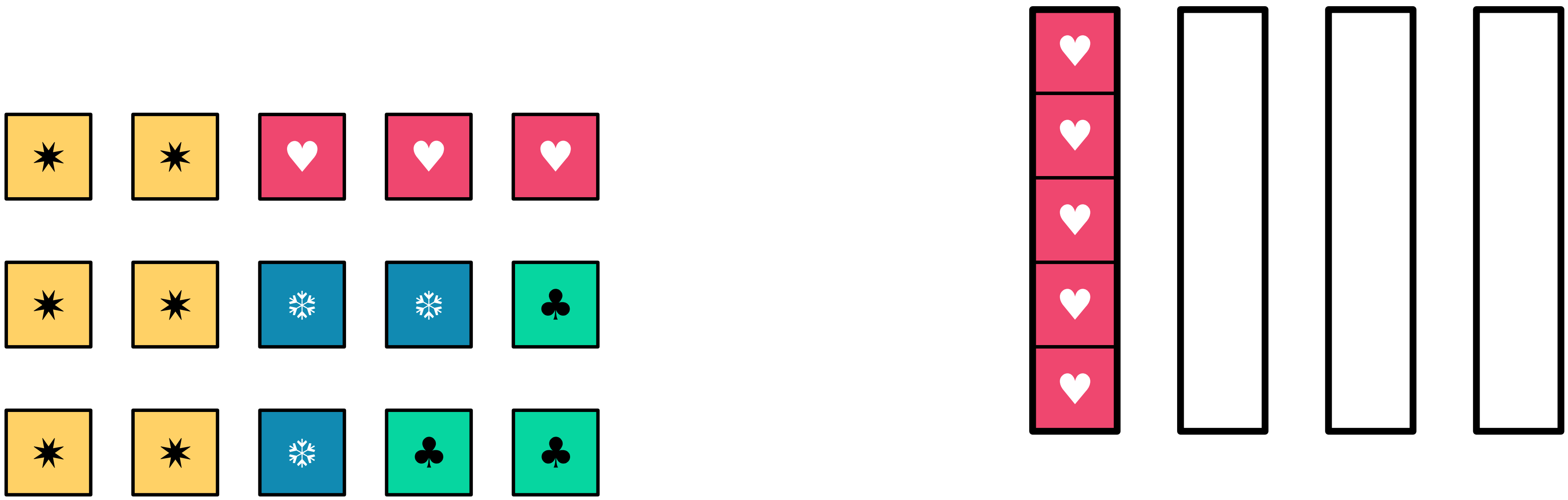

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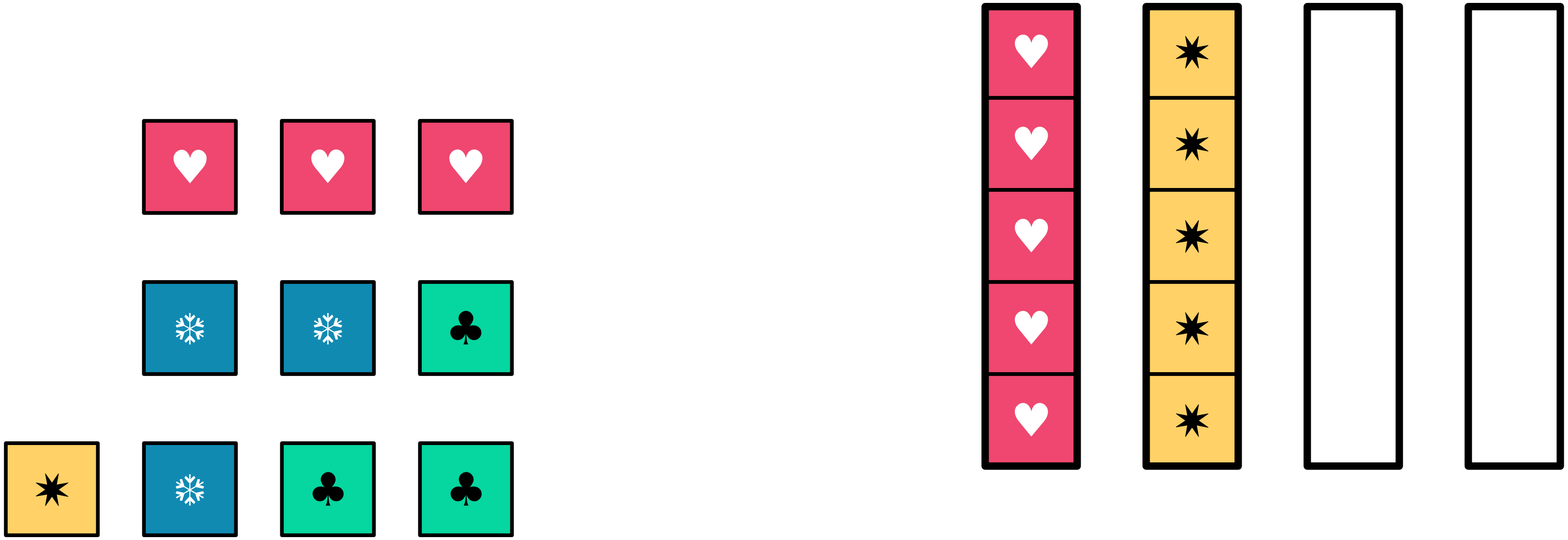

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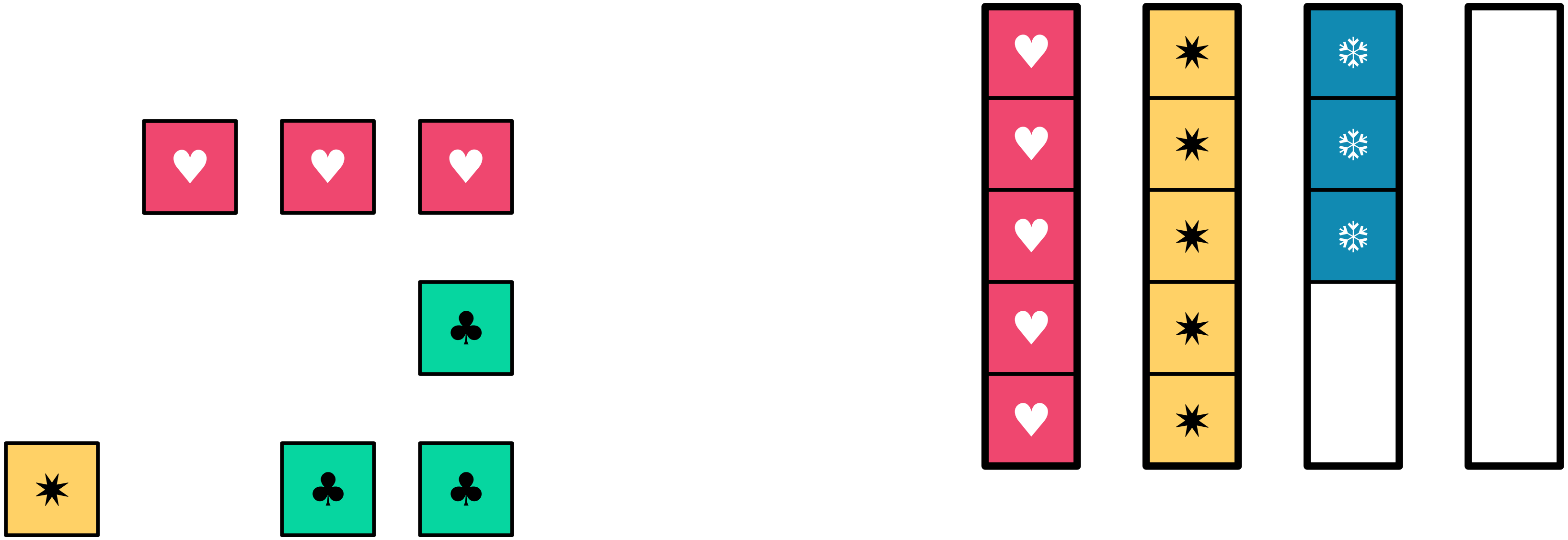

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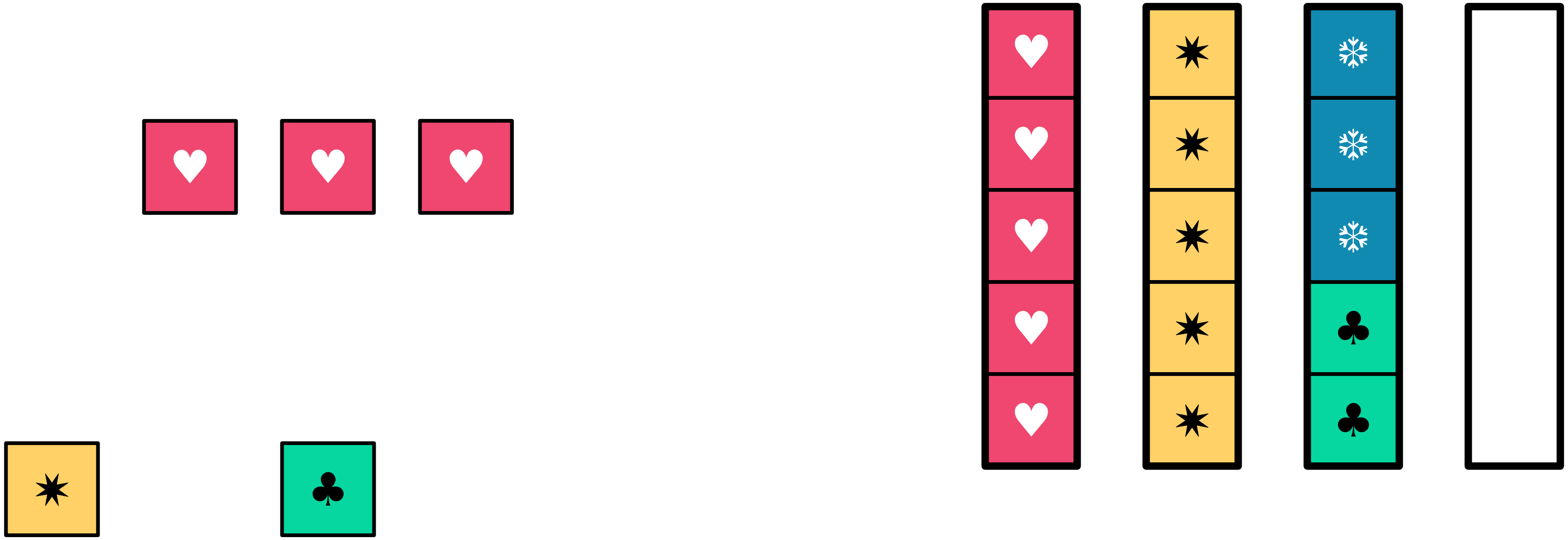

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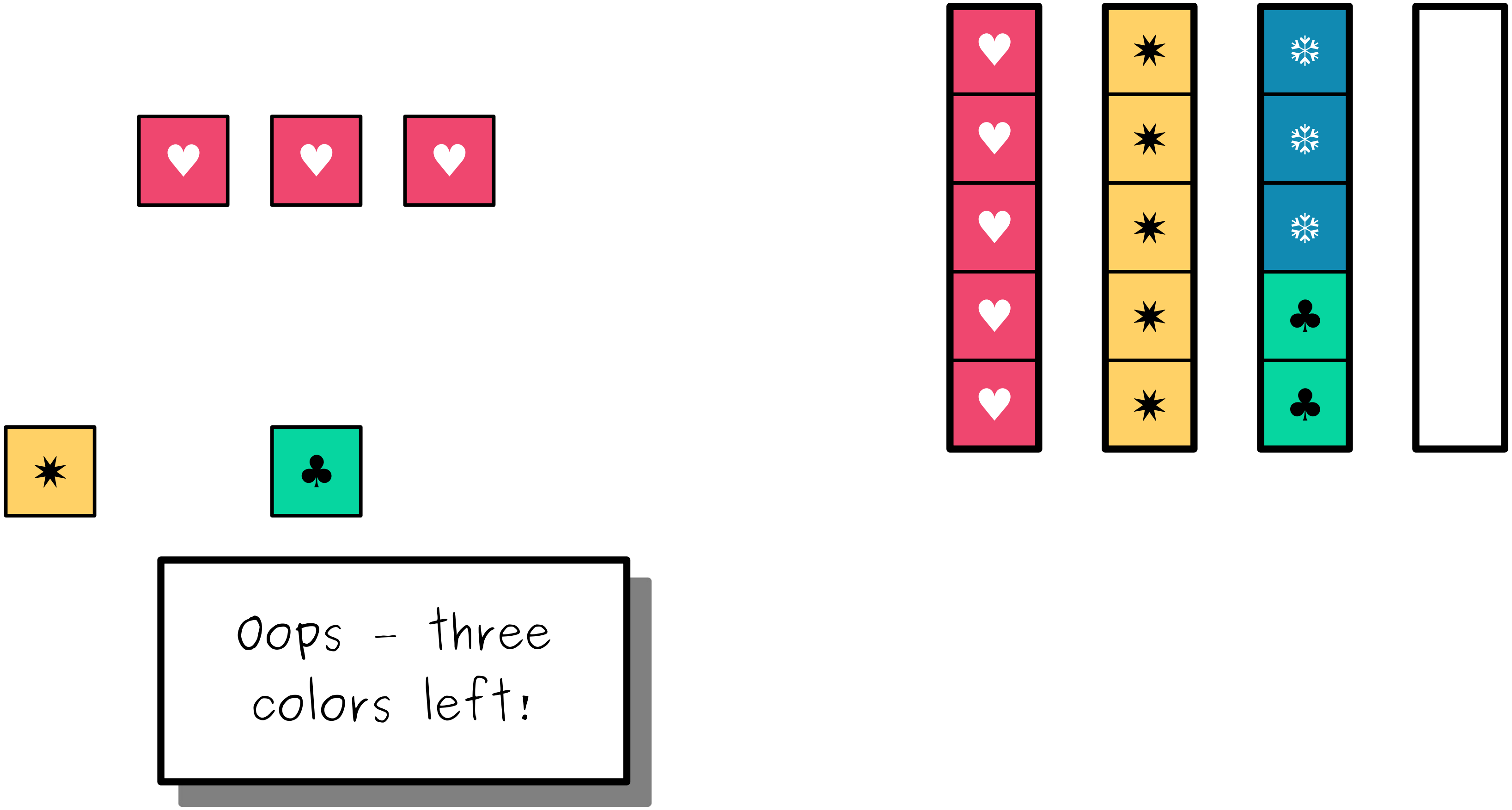

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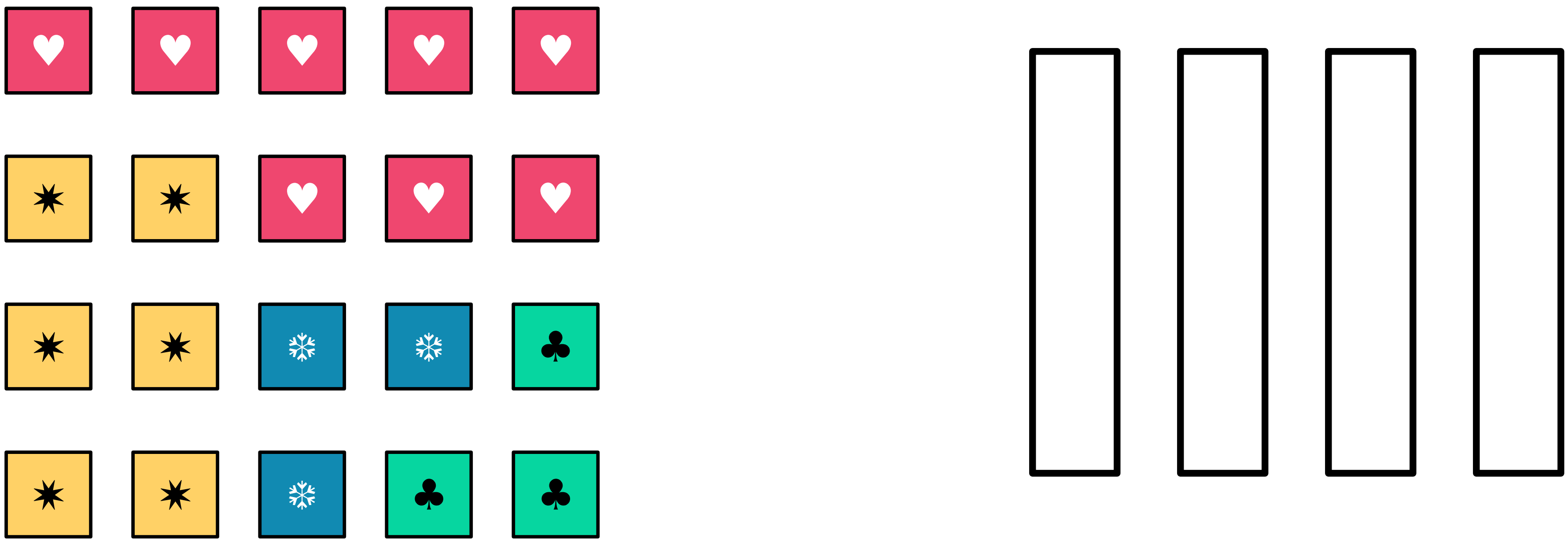

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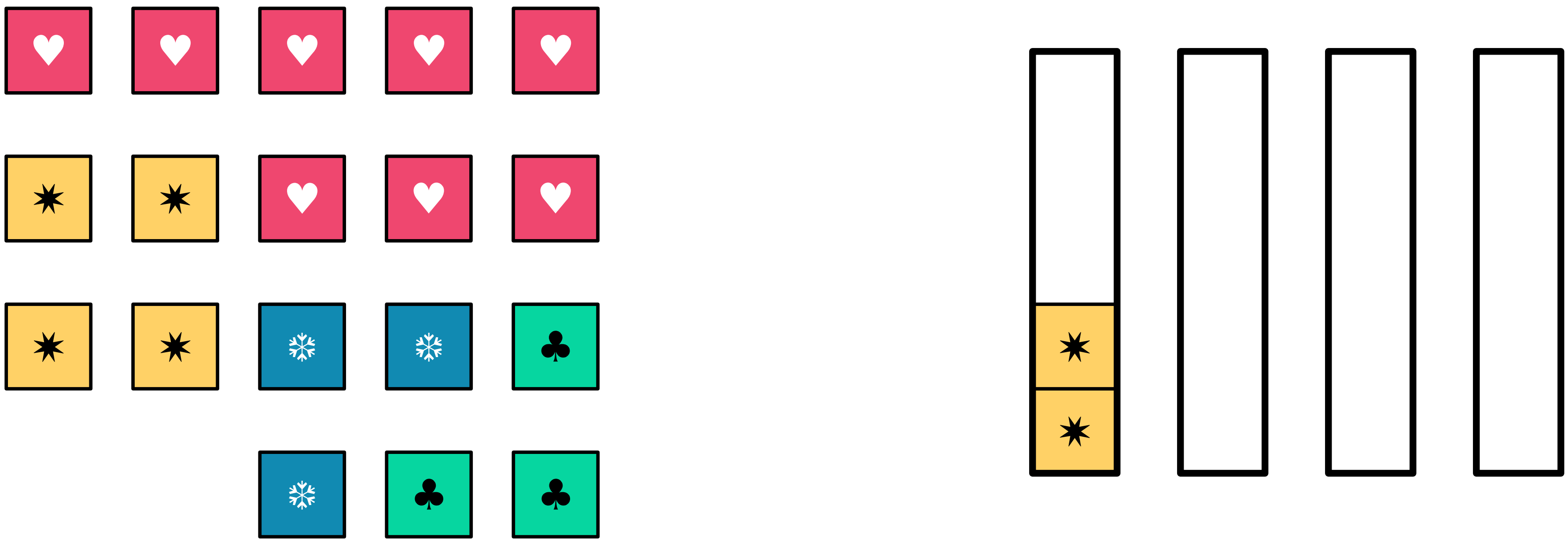

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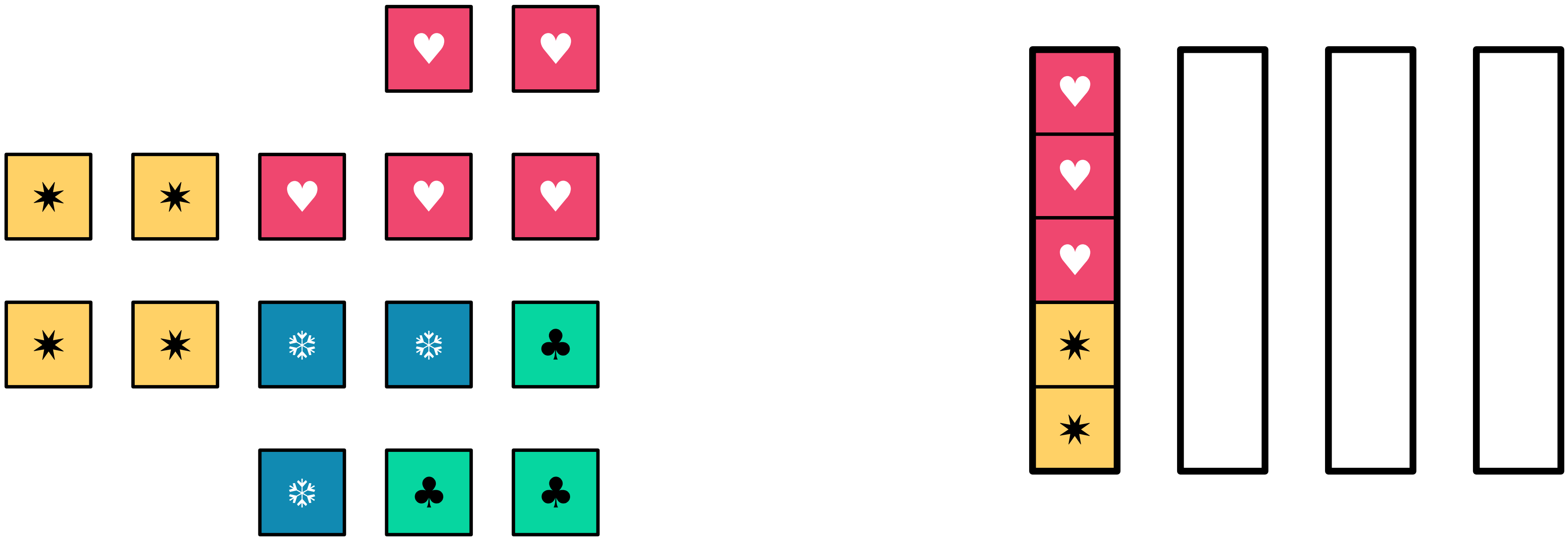

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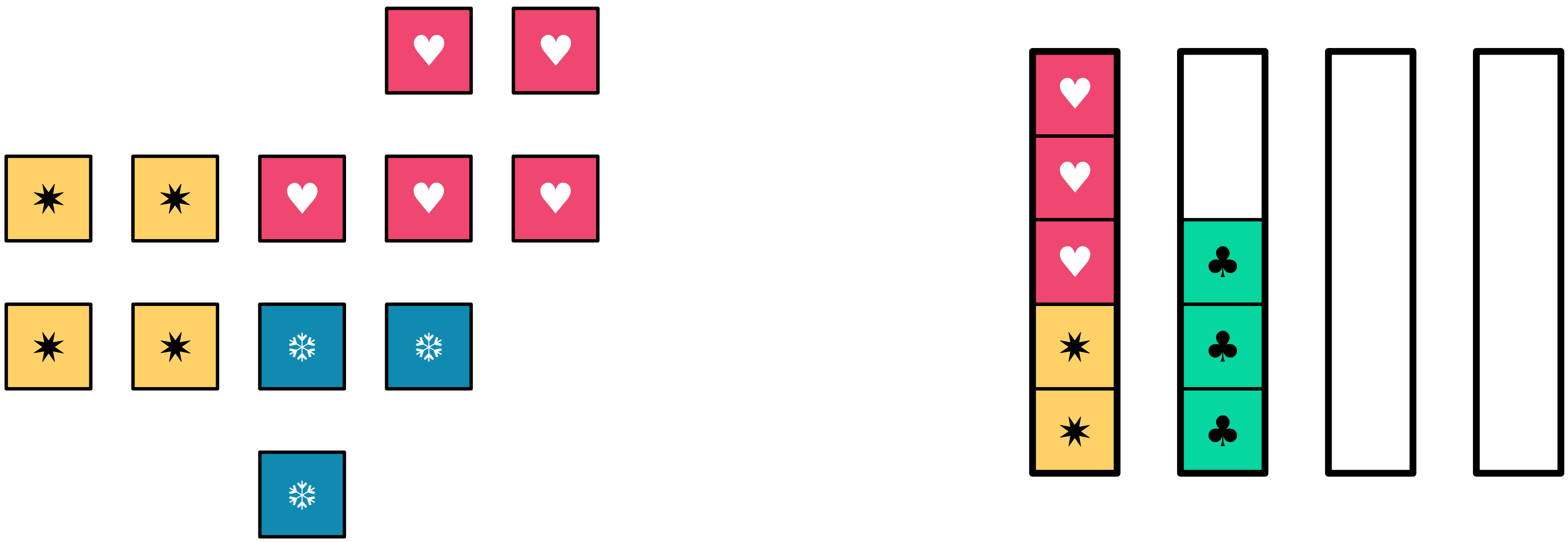

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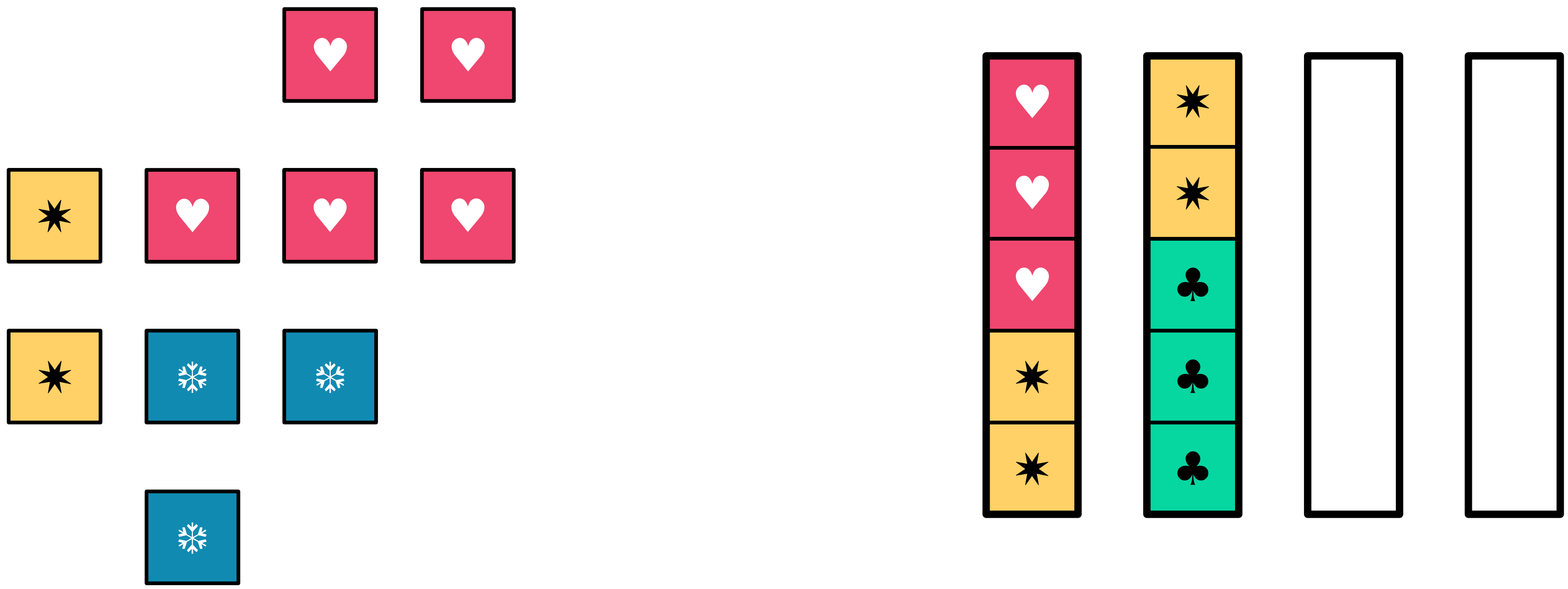

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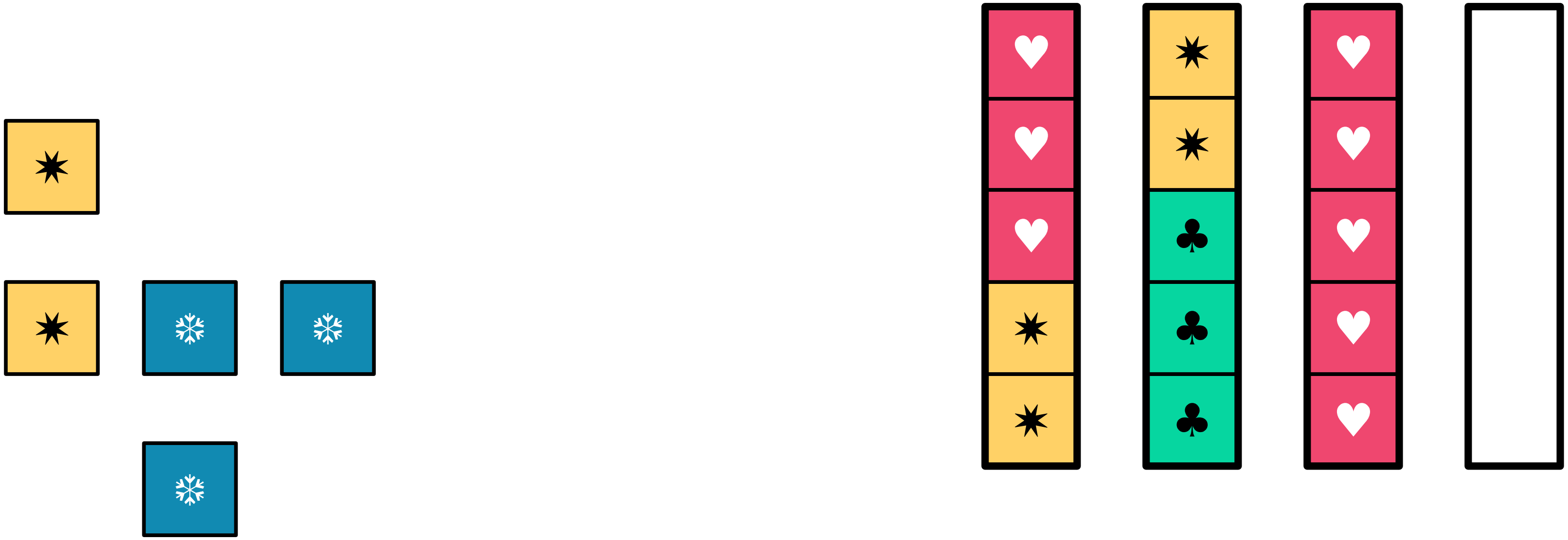

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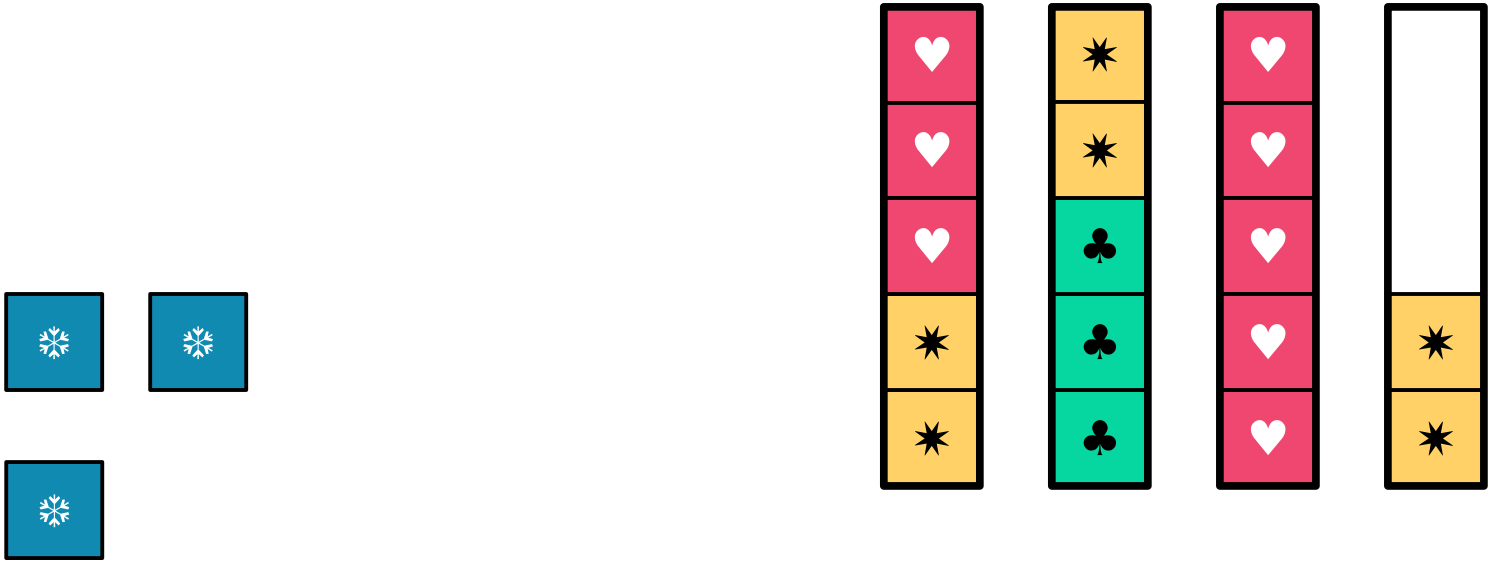

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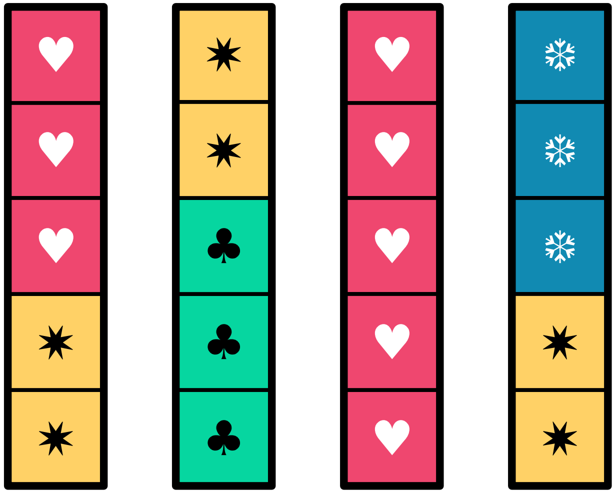

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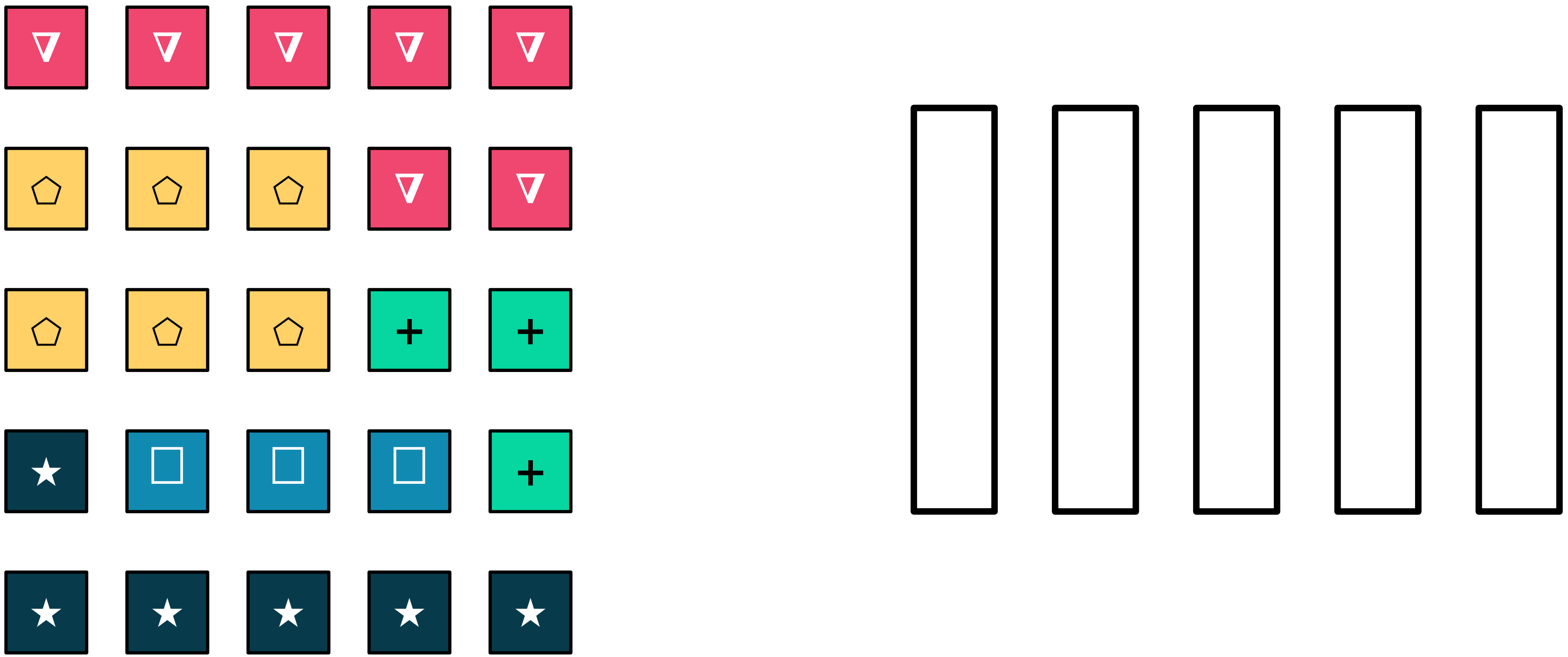

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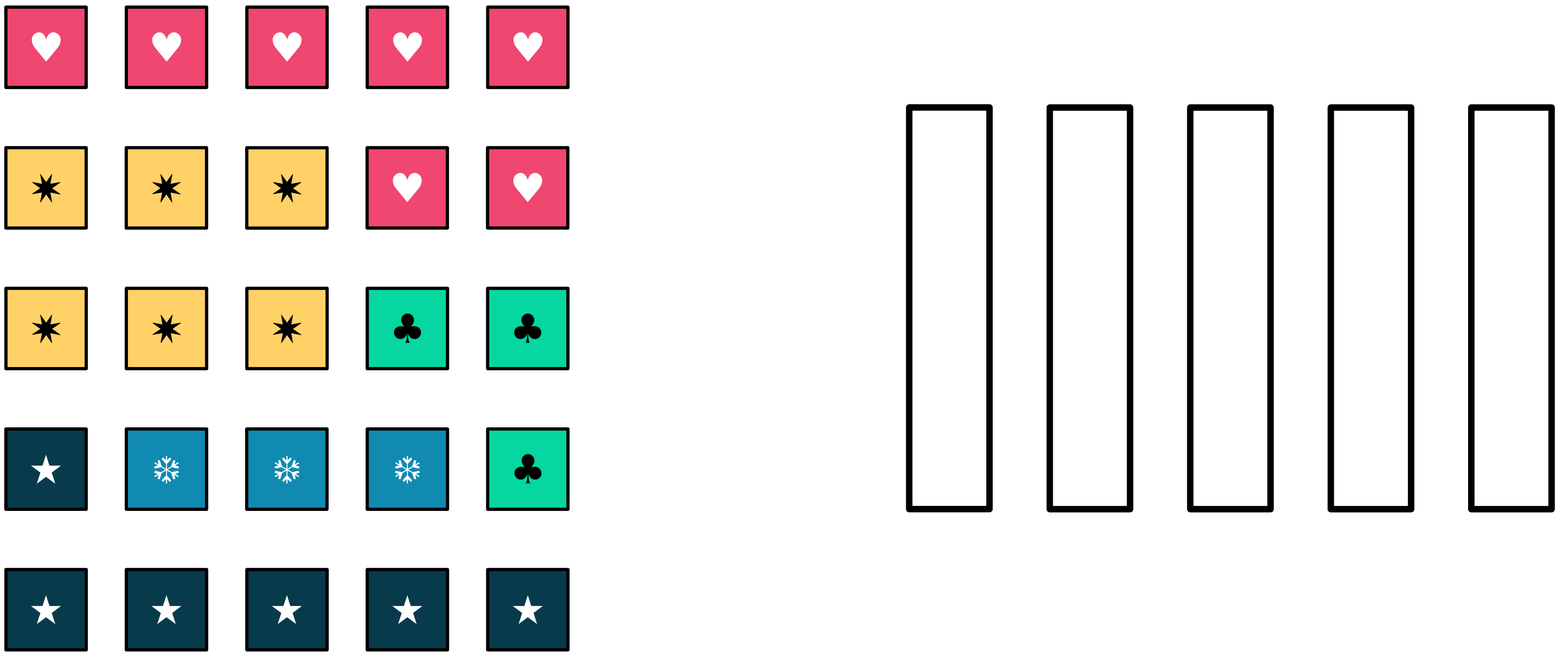

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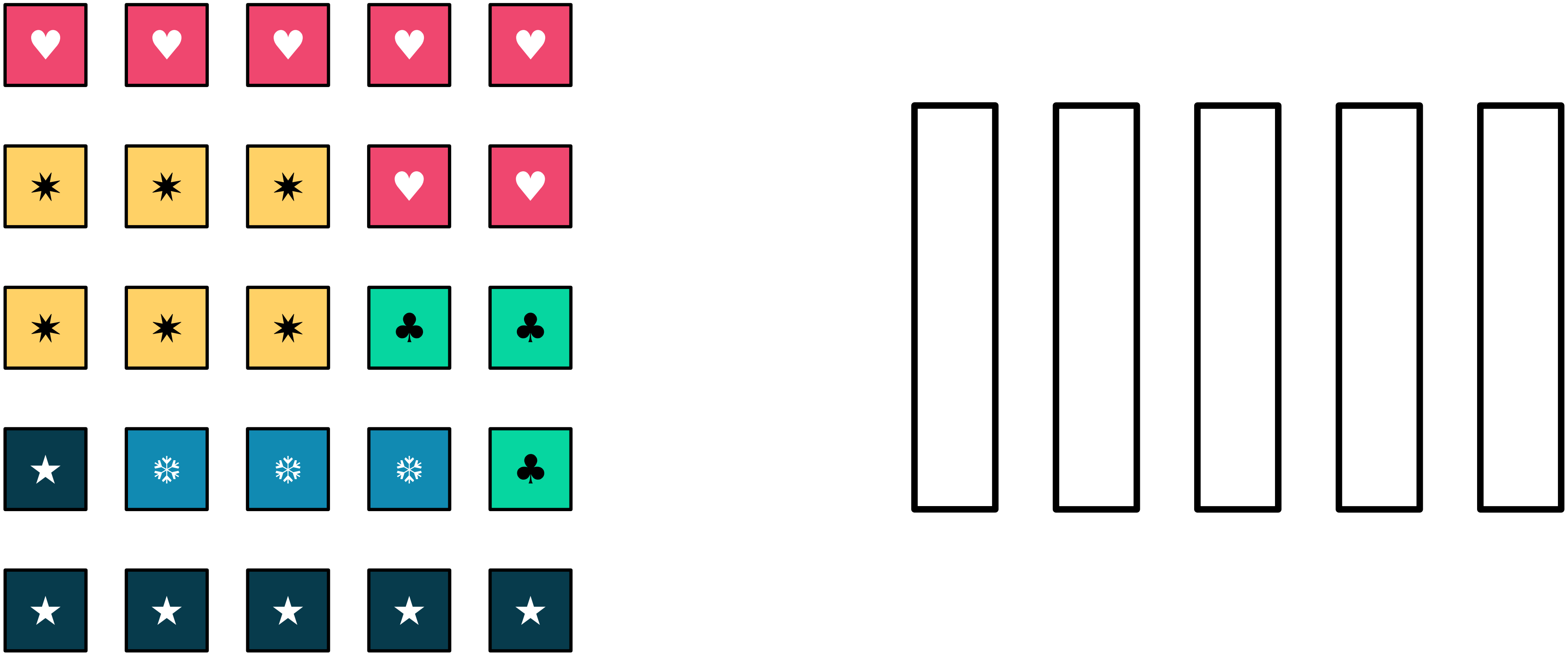

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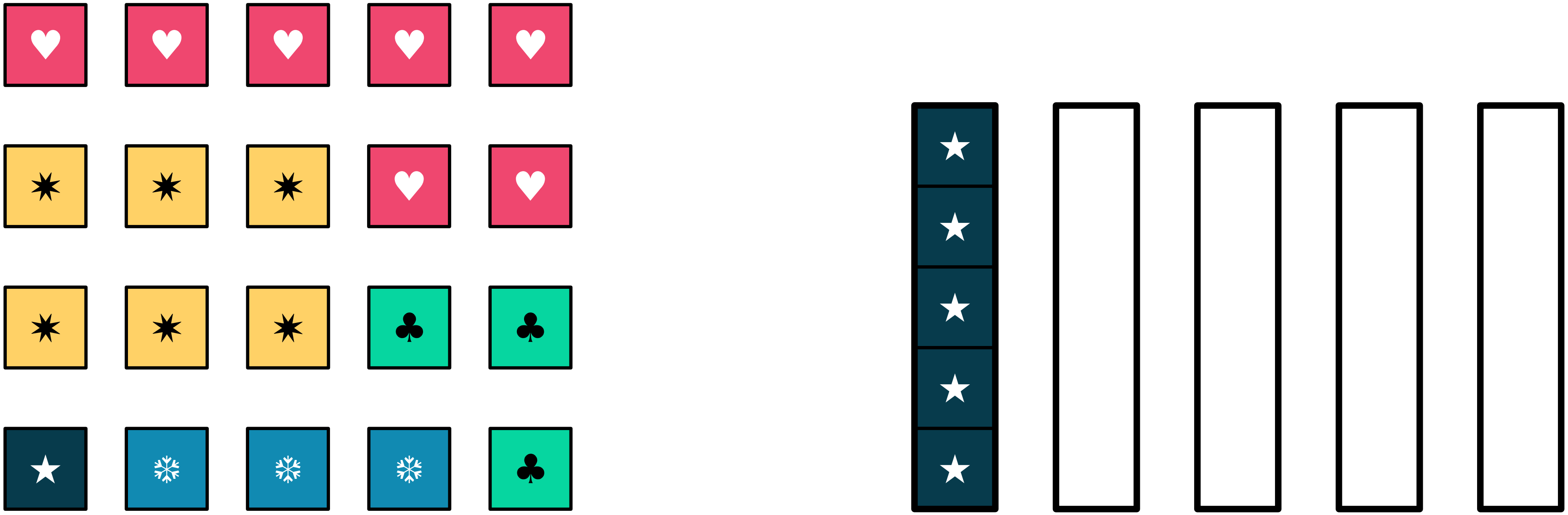

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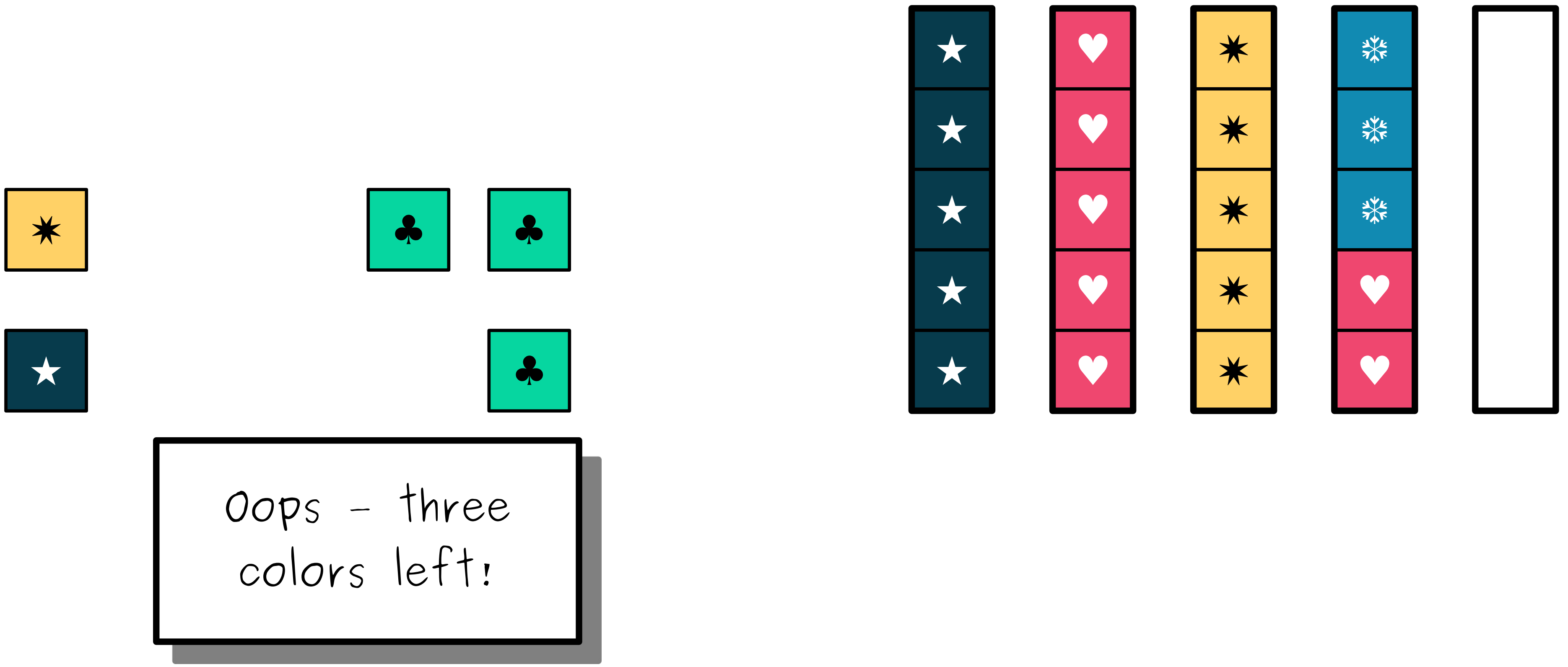

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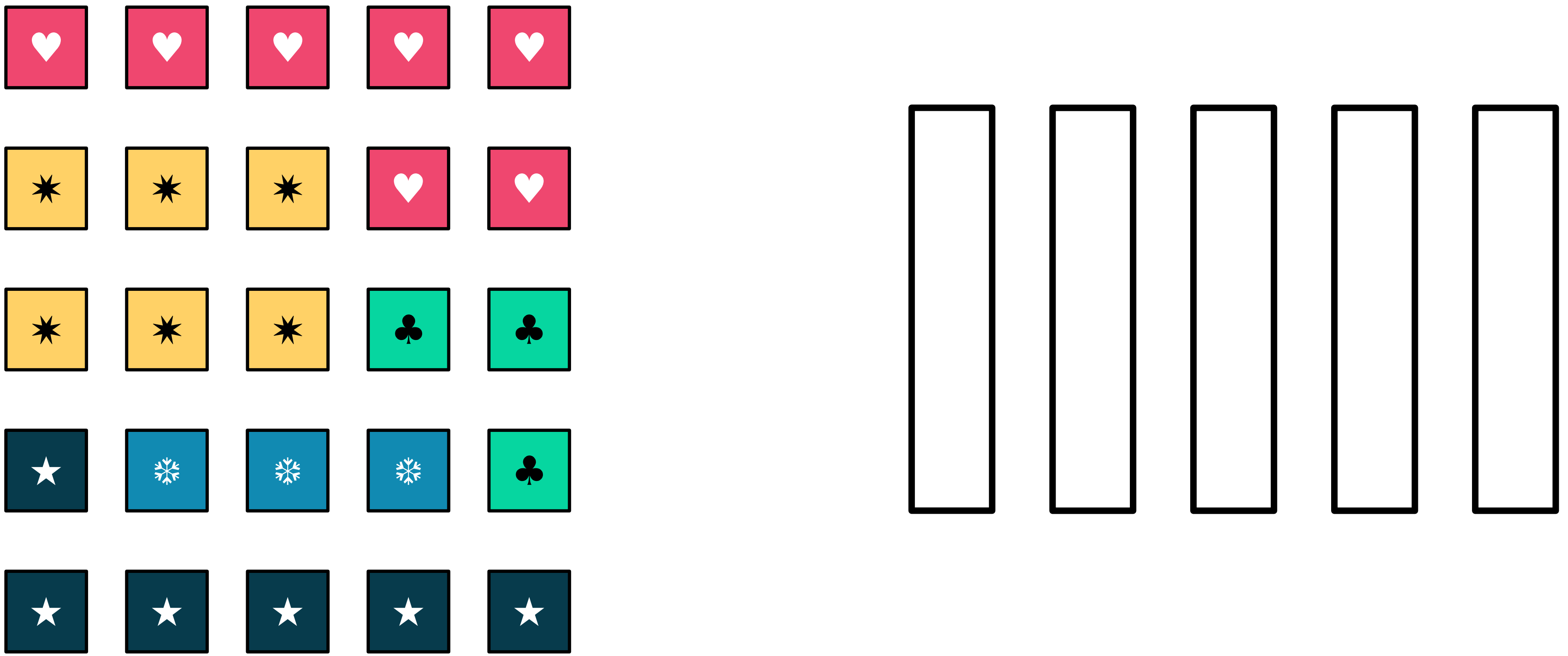

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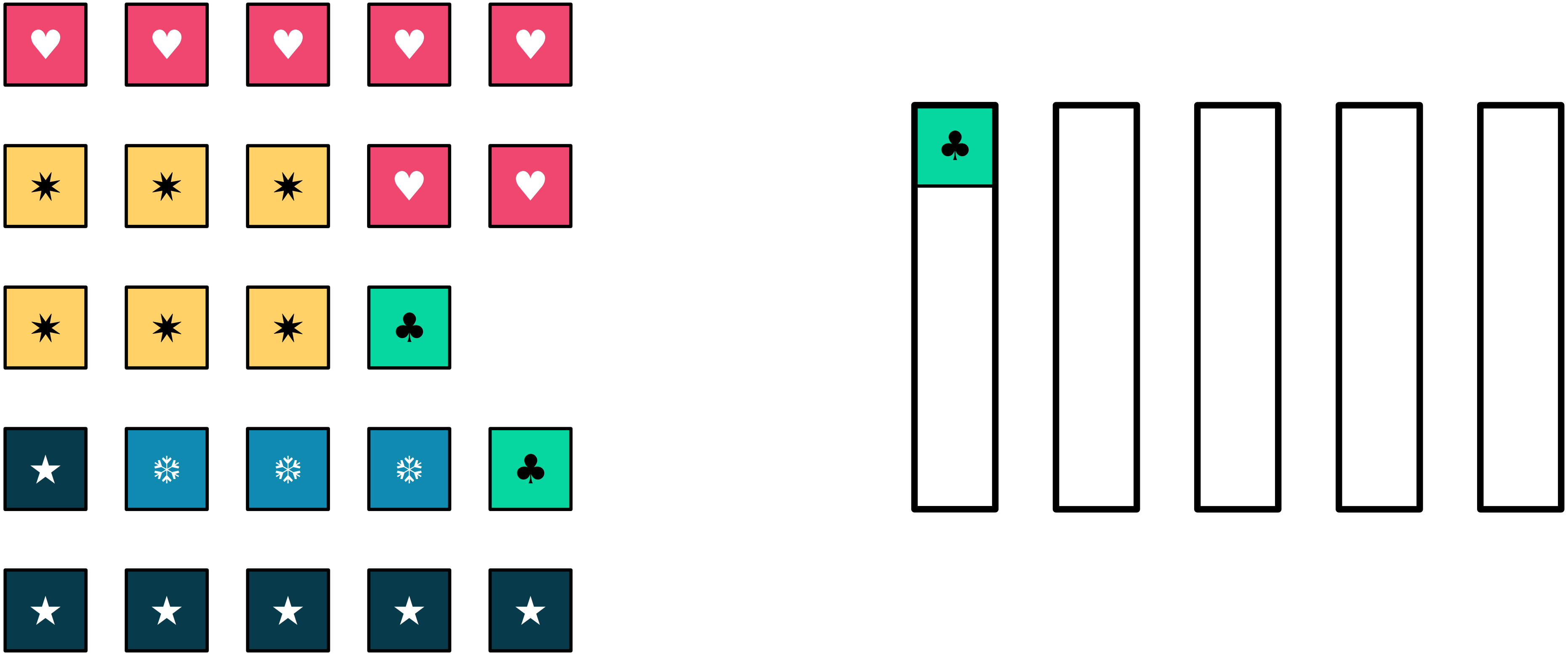

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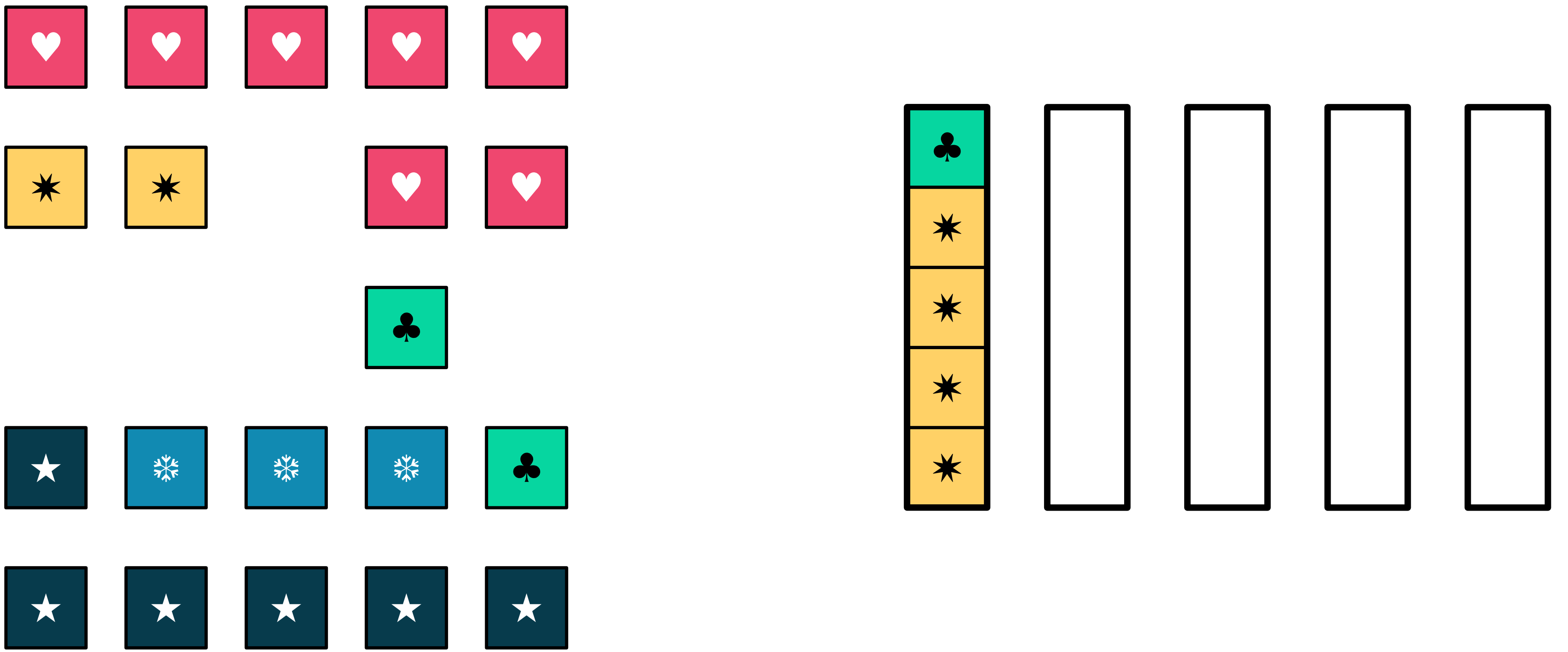

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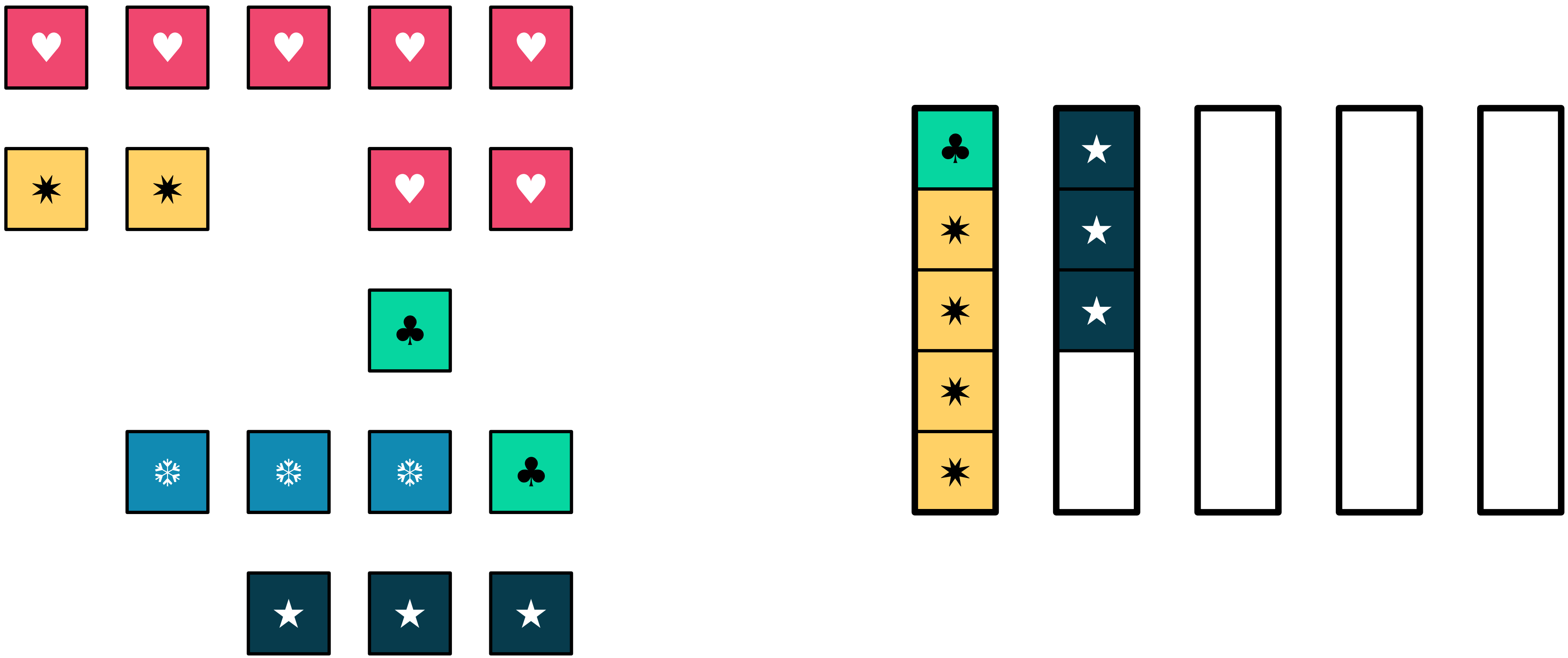

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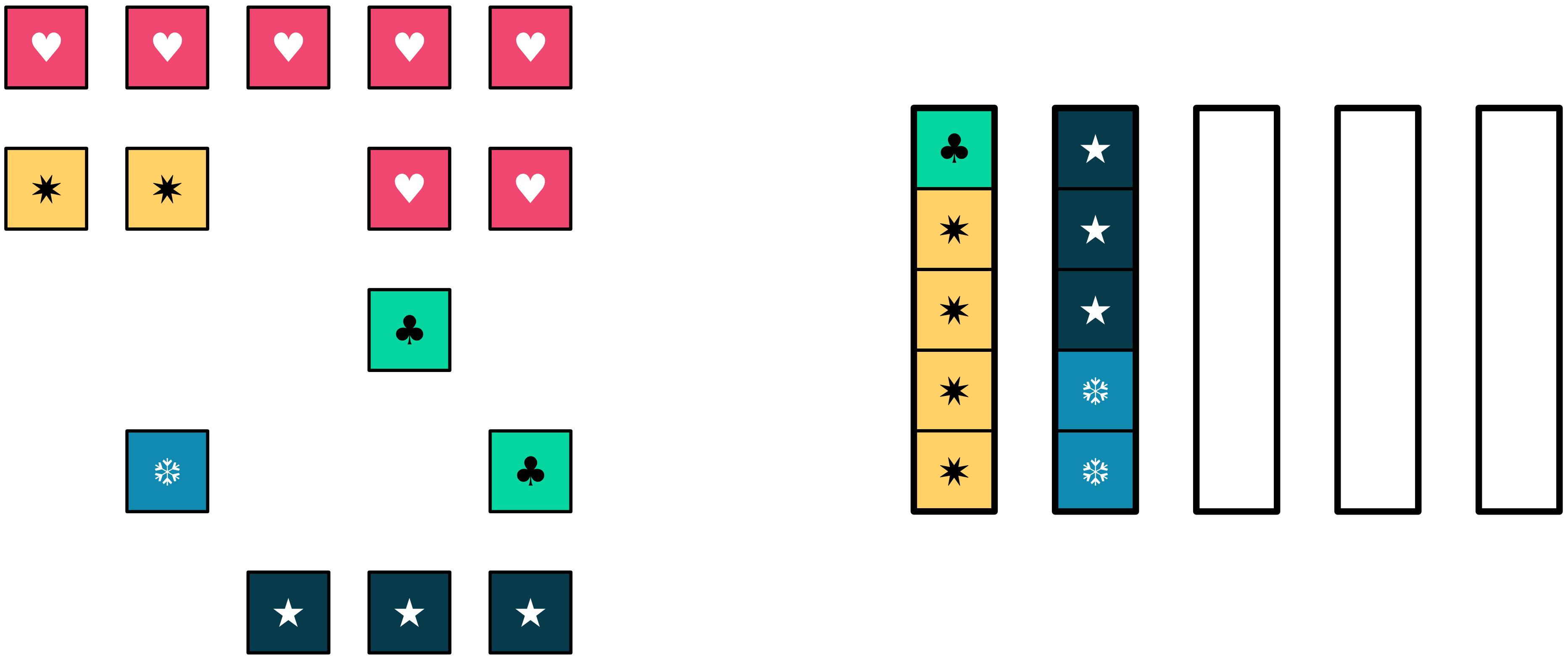

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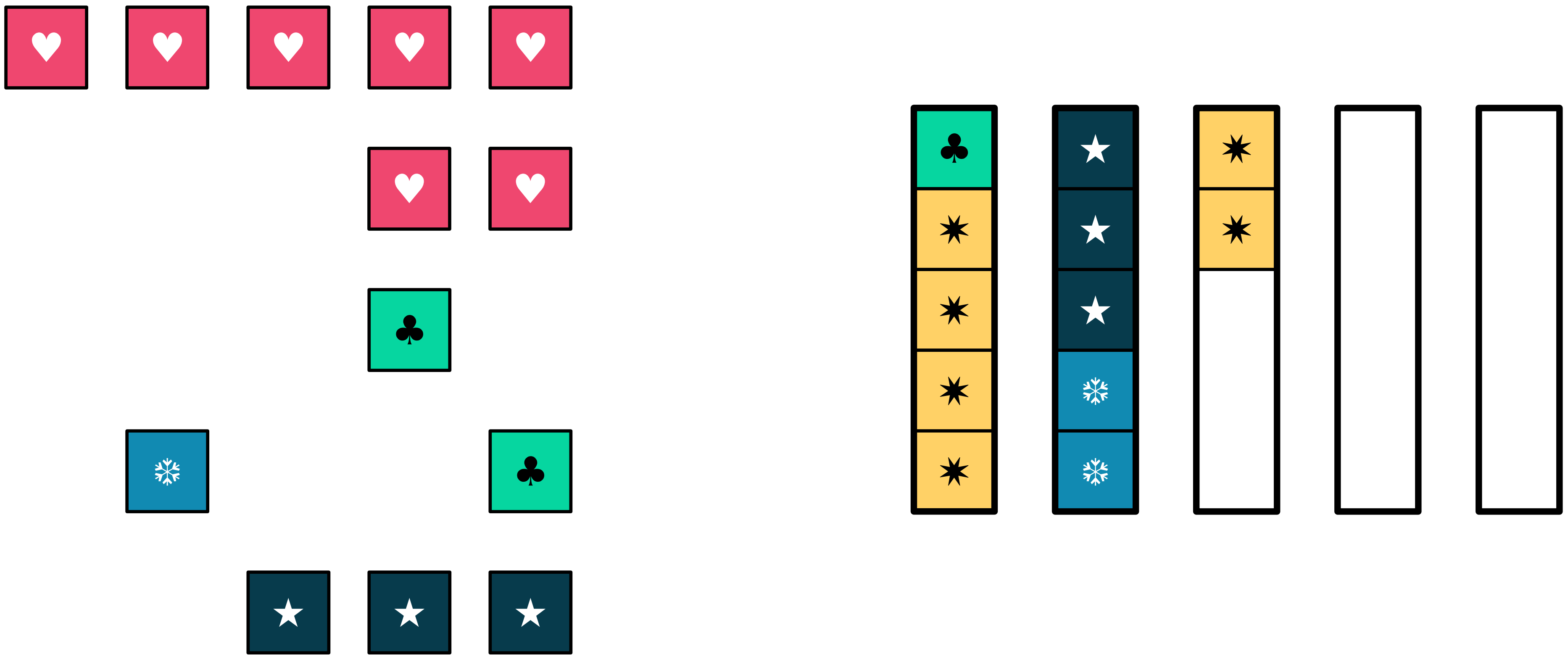

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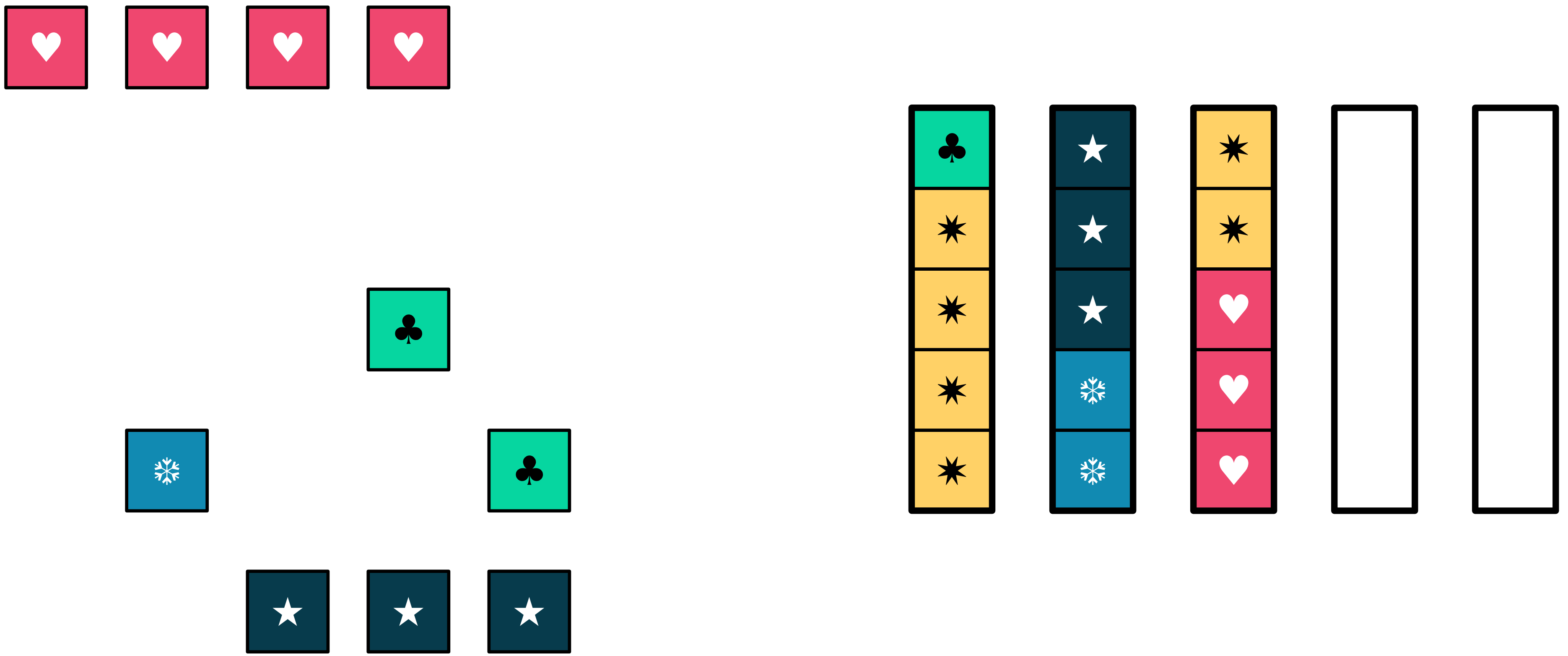

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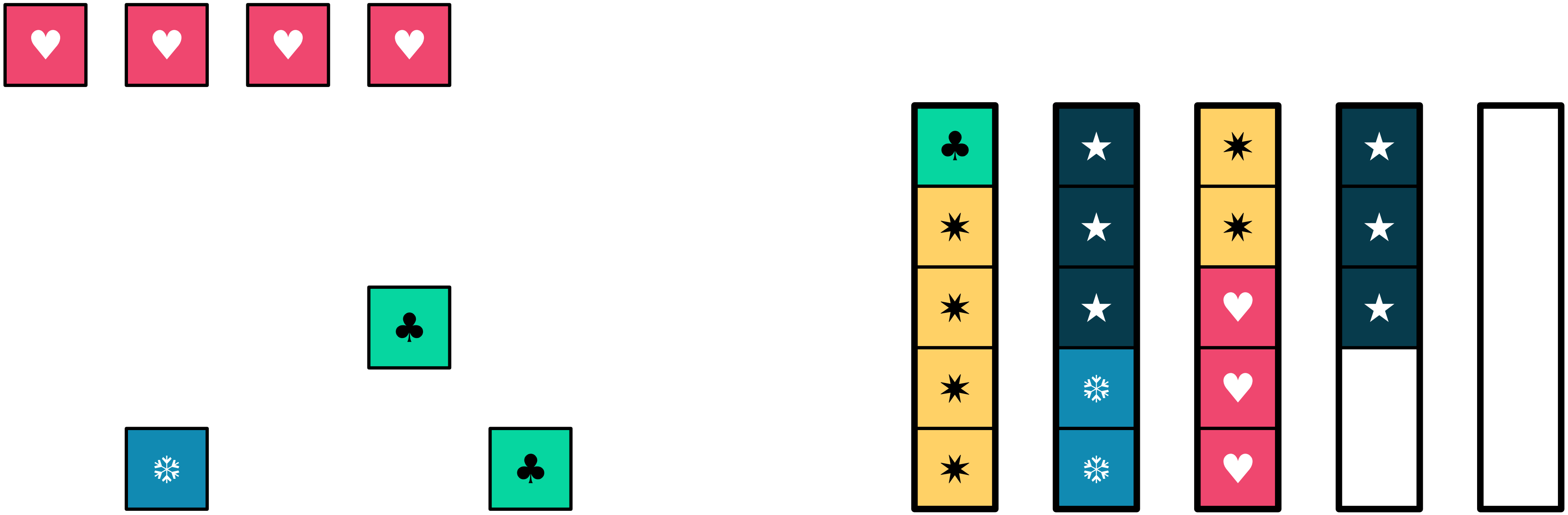

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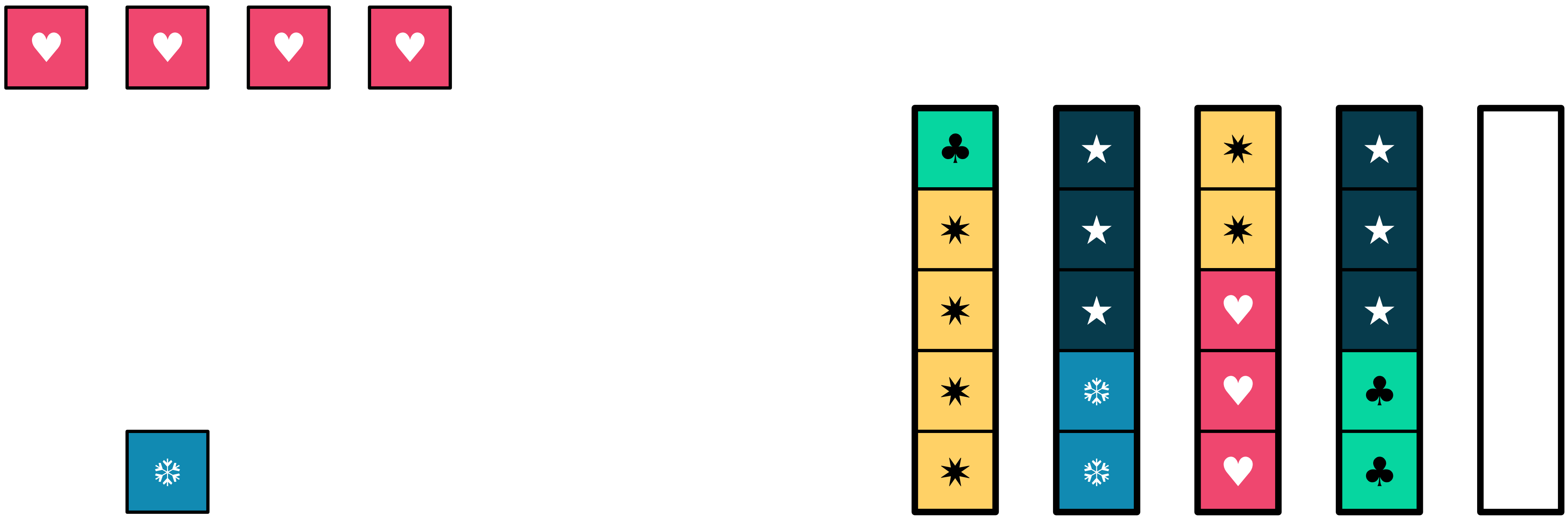

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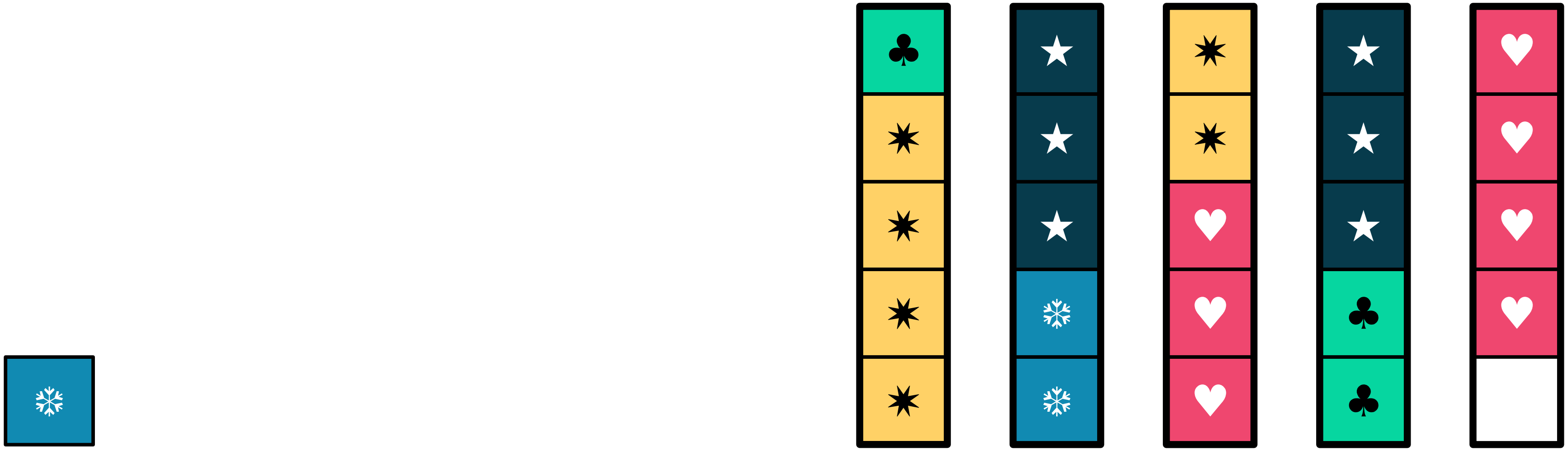

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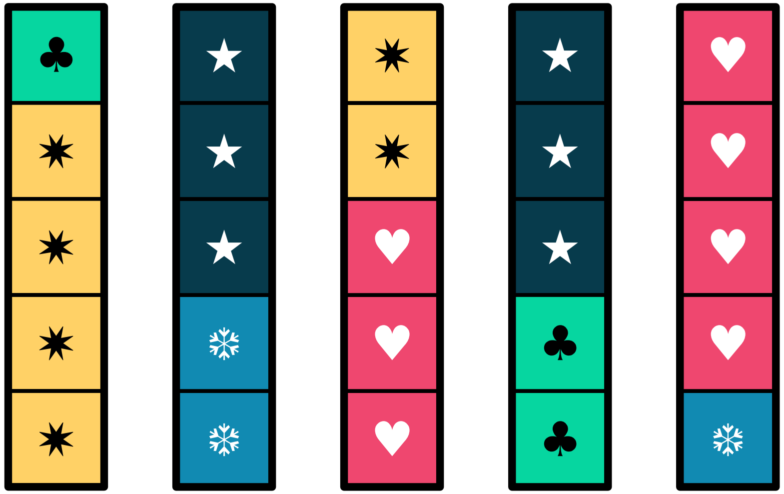

Here are 25 **potions** of 5 different **colors**.
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A ***good split*** of a group of $5n$ potions of n colors is a way of splitting them across shelves where each shelf has five potions with no more than two colors.

Theorem: For any group of $5n$ potions of n colors, there is a good split of those potions.

$P(n)$ is the statement "for any group of $5n$ potions of n colors, there exists a good split of those potions."

Basically, do nothing! ➜ $P(0)$

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.

P(n) is the statement "for any group of $5n$ potions of n colors, there exists a good split of those potions."

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

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What kind of quantifier is embedded in our predicate, P(n)?

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$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

if

$P(k)$

then

$P(k+1)$

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$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

if

for every group of $5k$ potions of k colors, there's a good split of those potions.

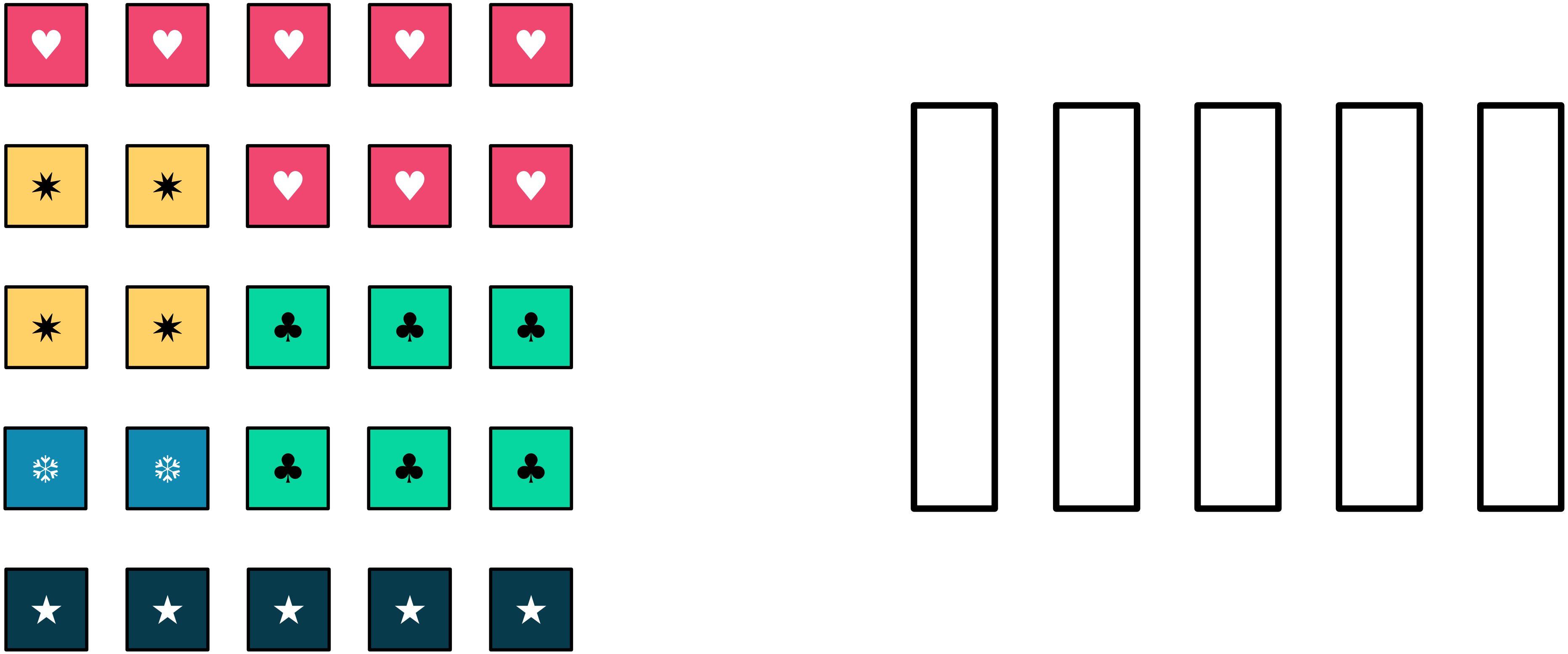
then

for every group of $5k+5$ potions of $k+1$ colors, there's a good split of those potions.

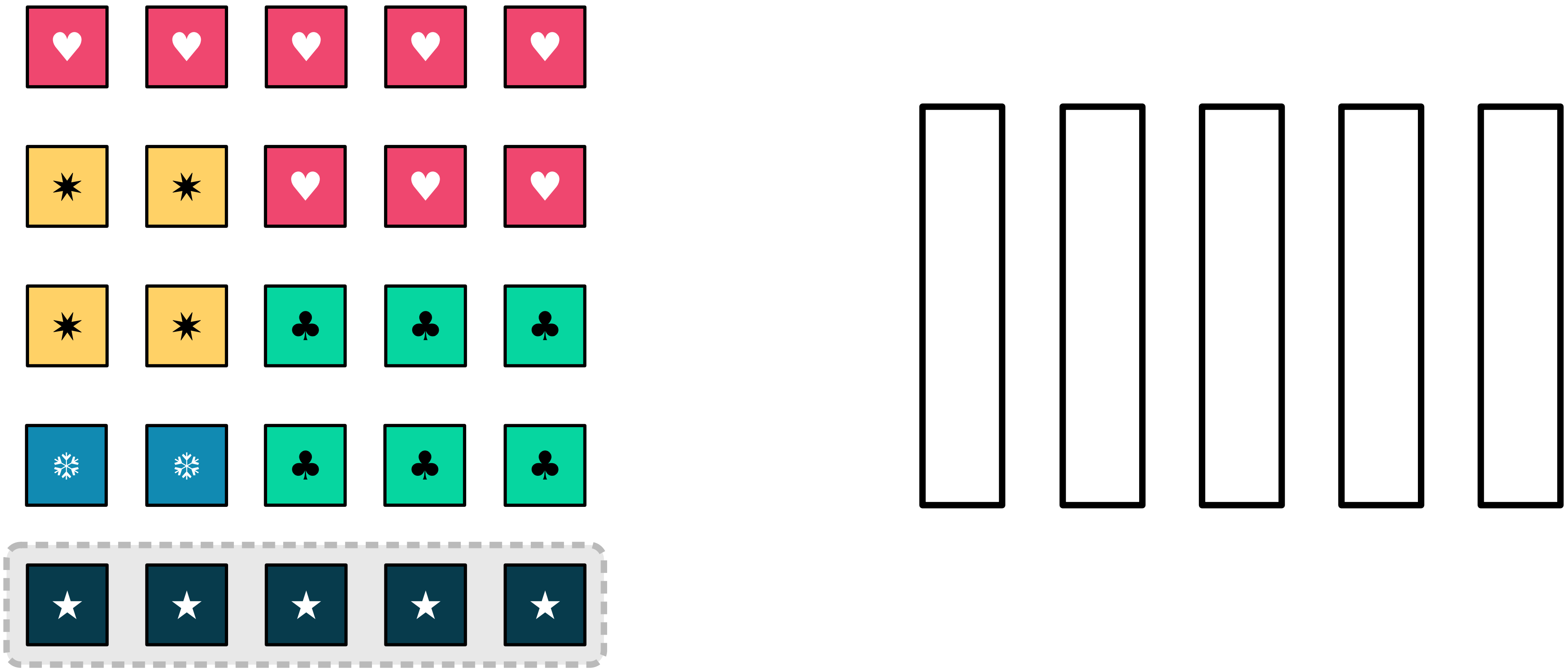
Assume a universal

Prove a universal

Theorem: For any group of $5n$ potions of n different, colors there exists a good split of those potions.



Idea: Begin with $5k+5$ potions and $k+1$ colors.
Find a way to remove five potions and one color.



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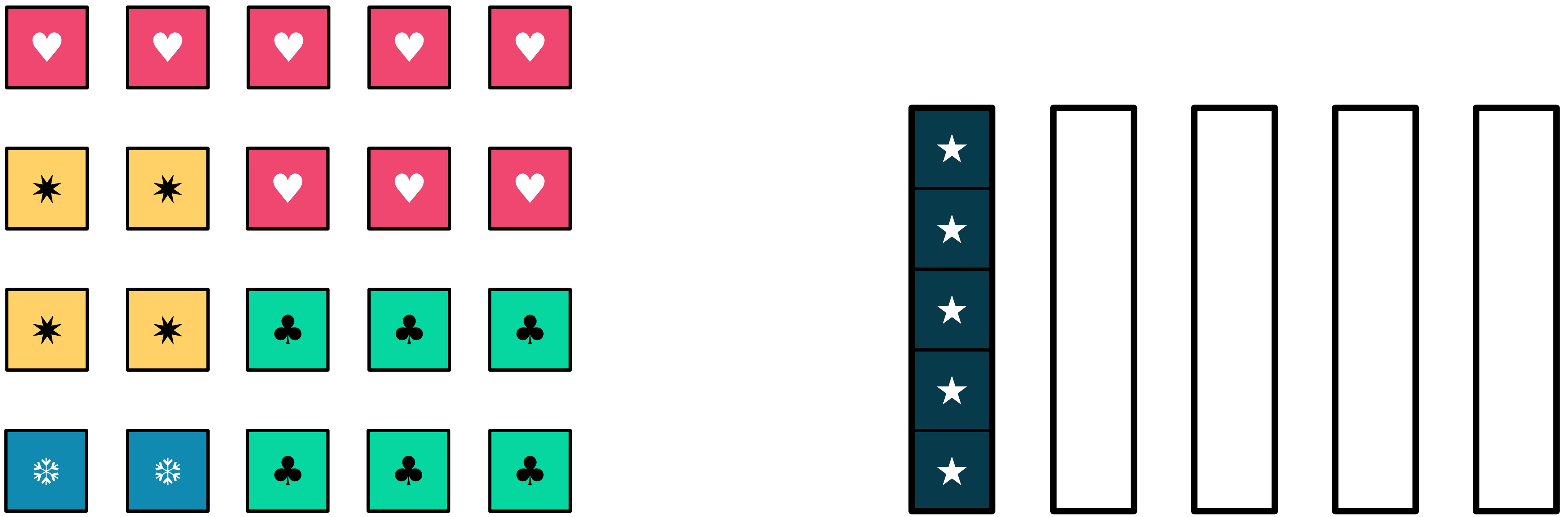

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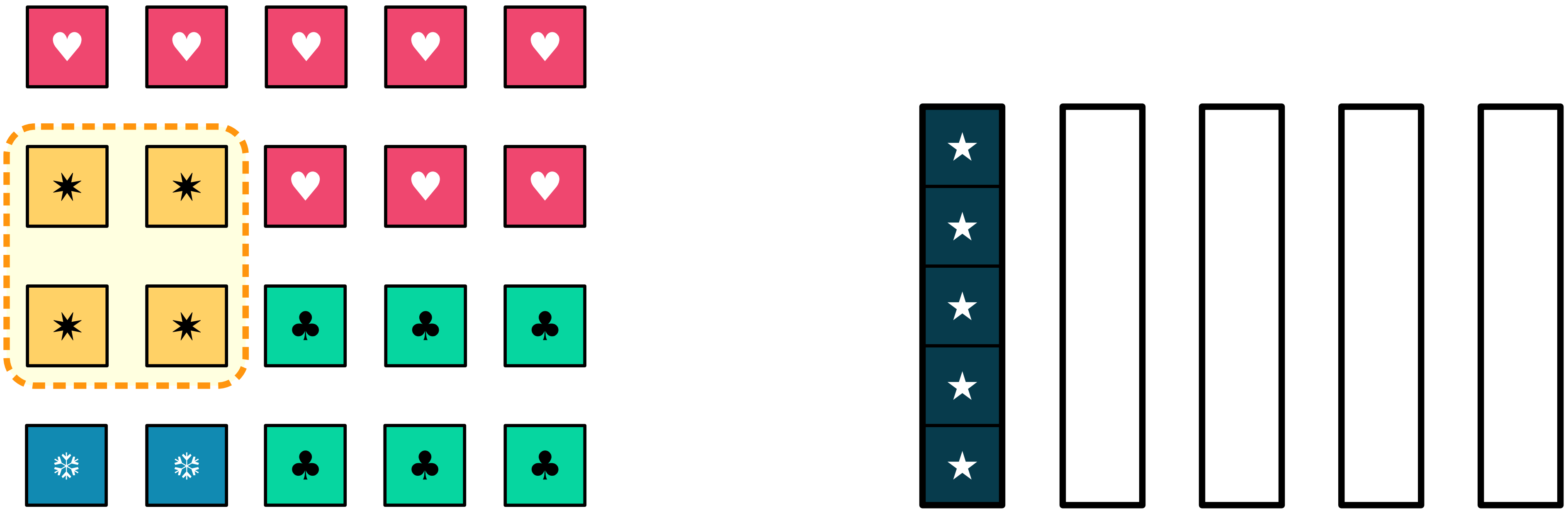

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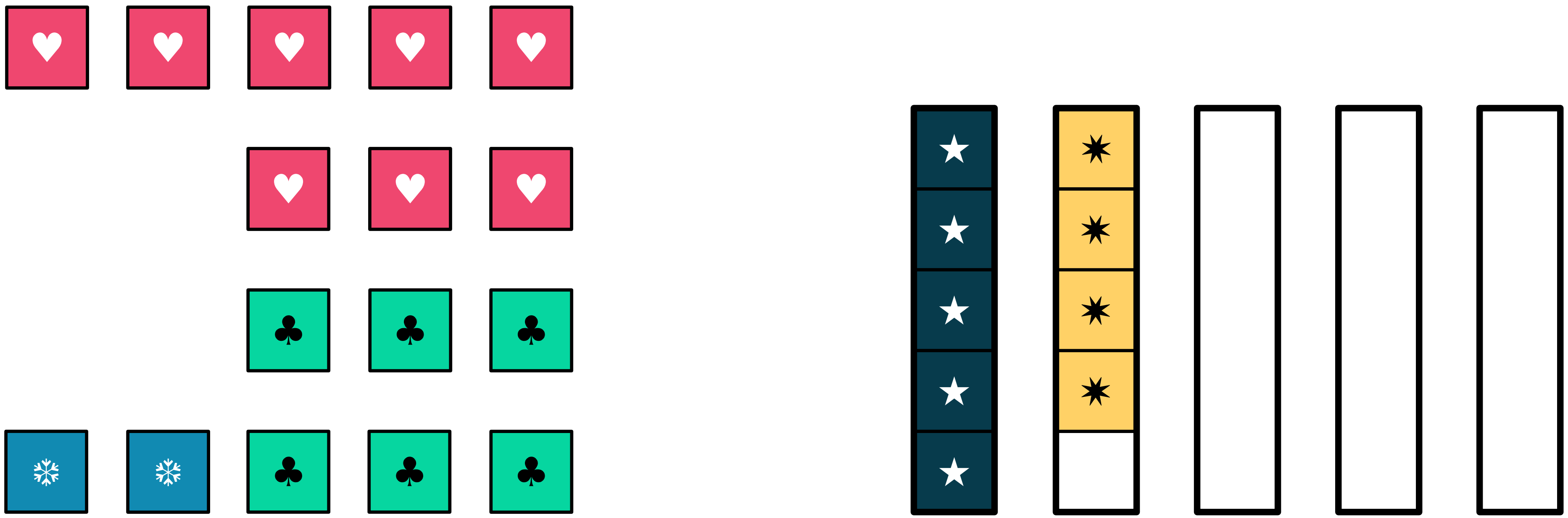

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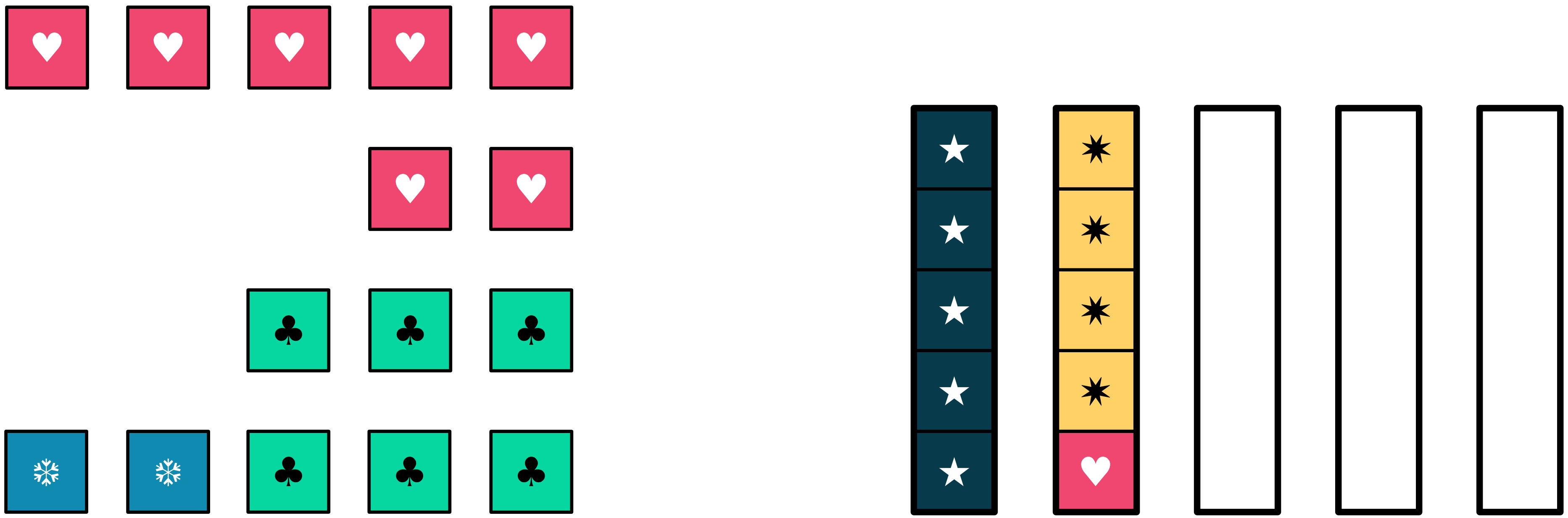

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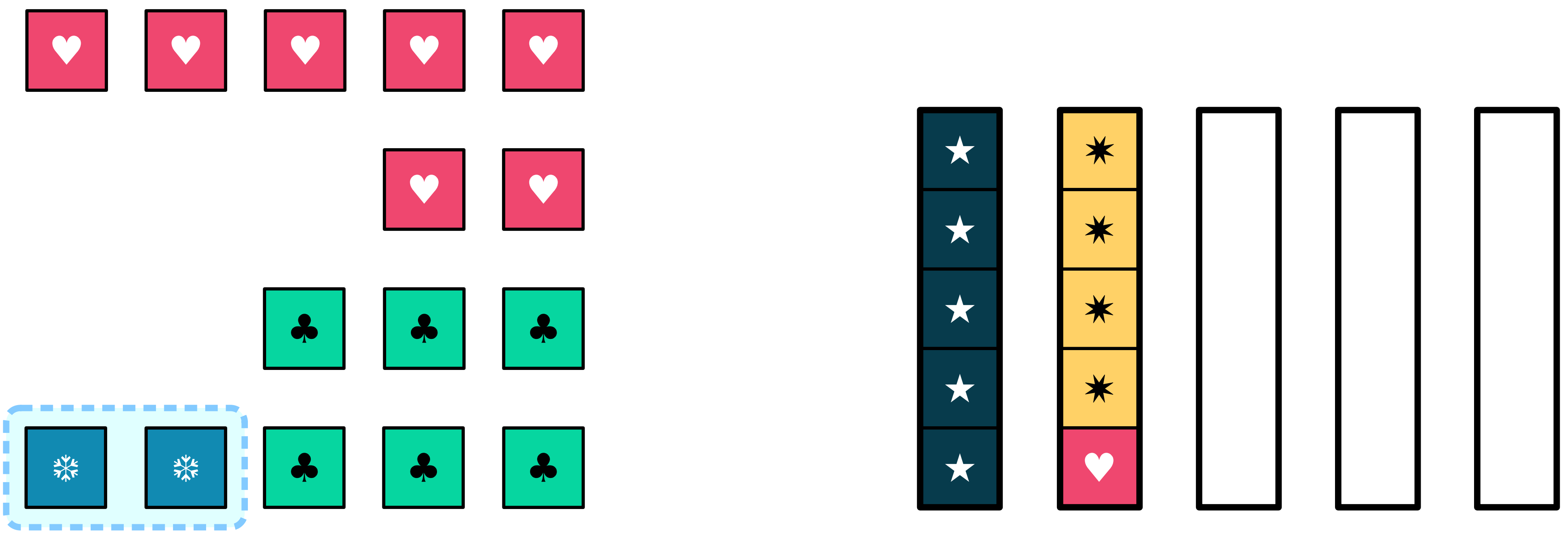

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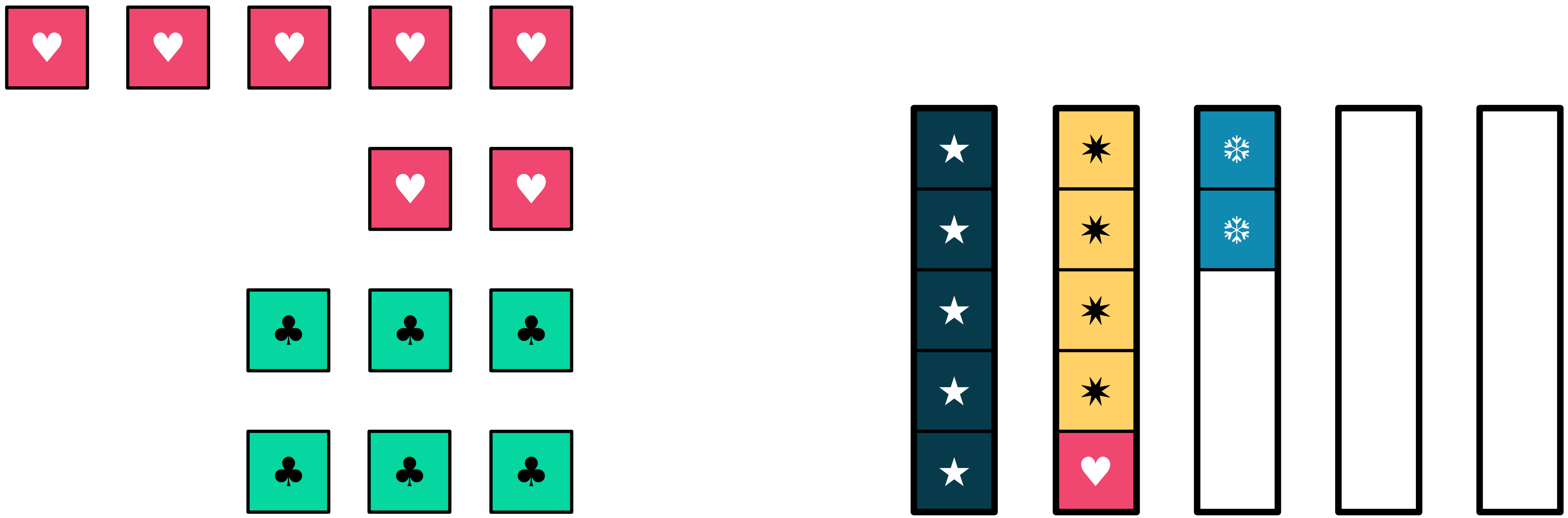

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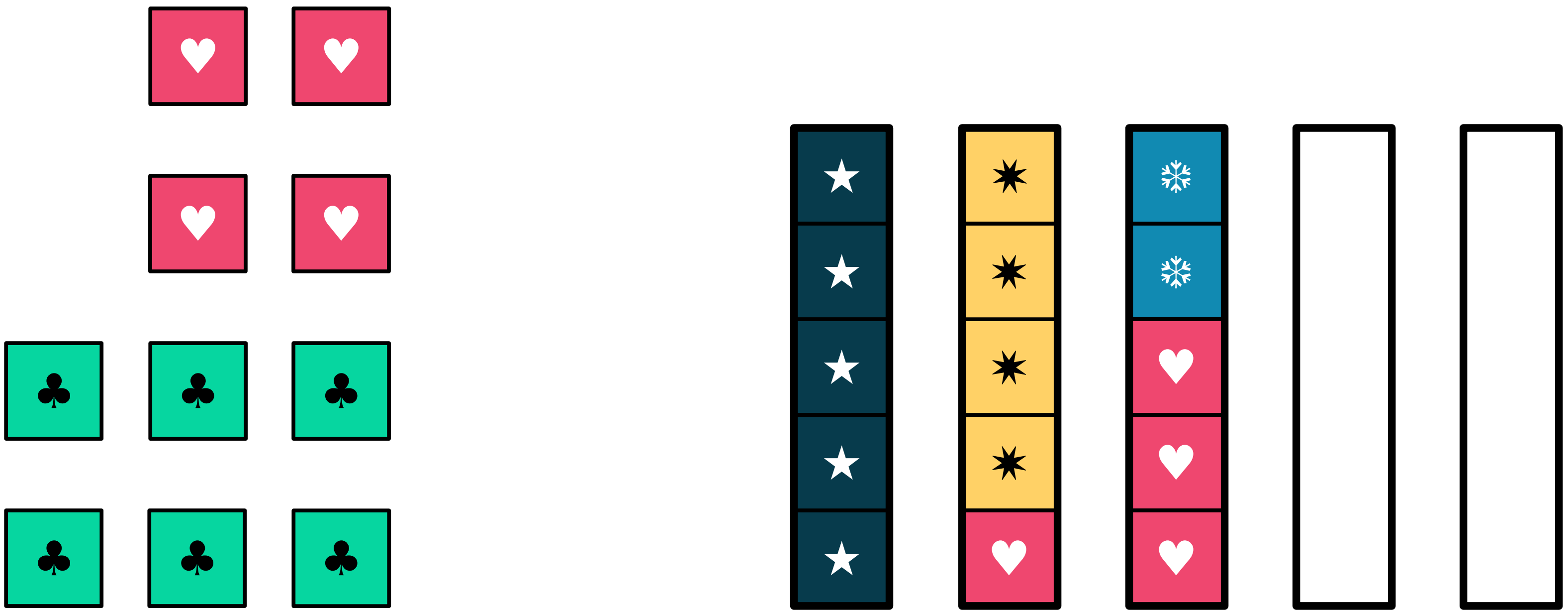

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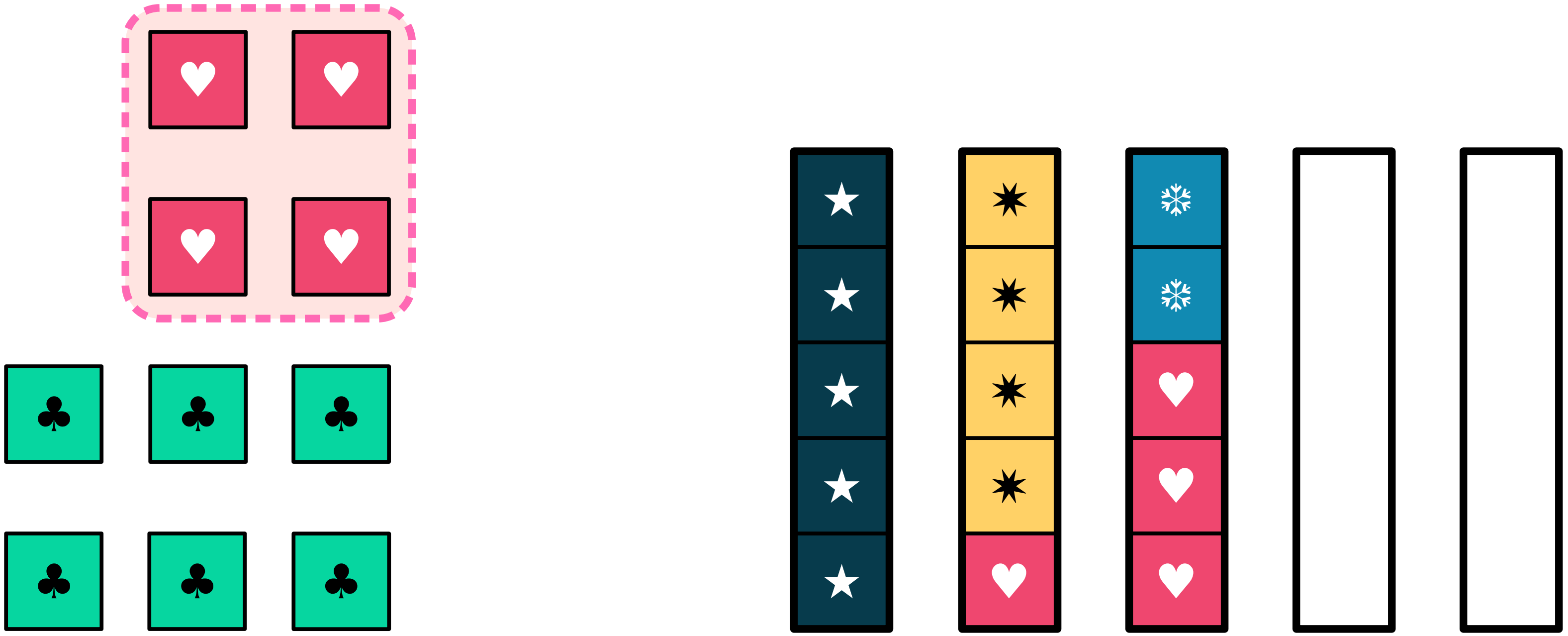

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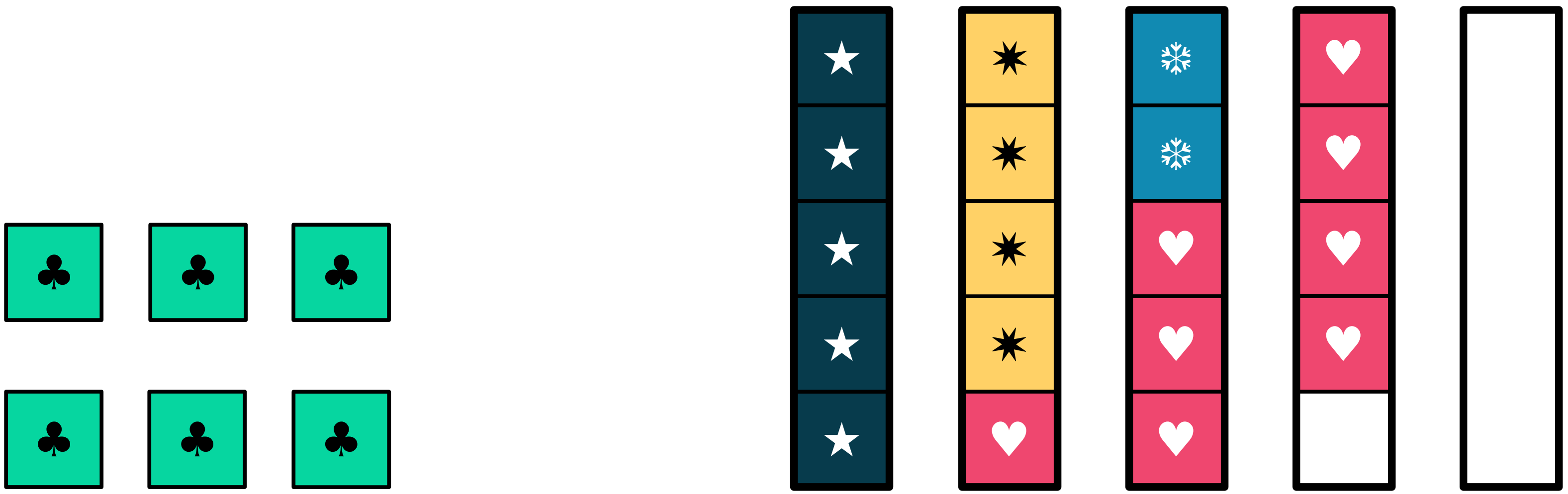
Find a way to remove five potions and one color.



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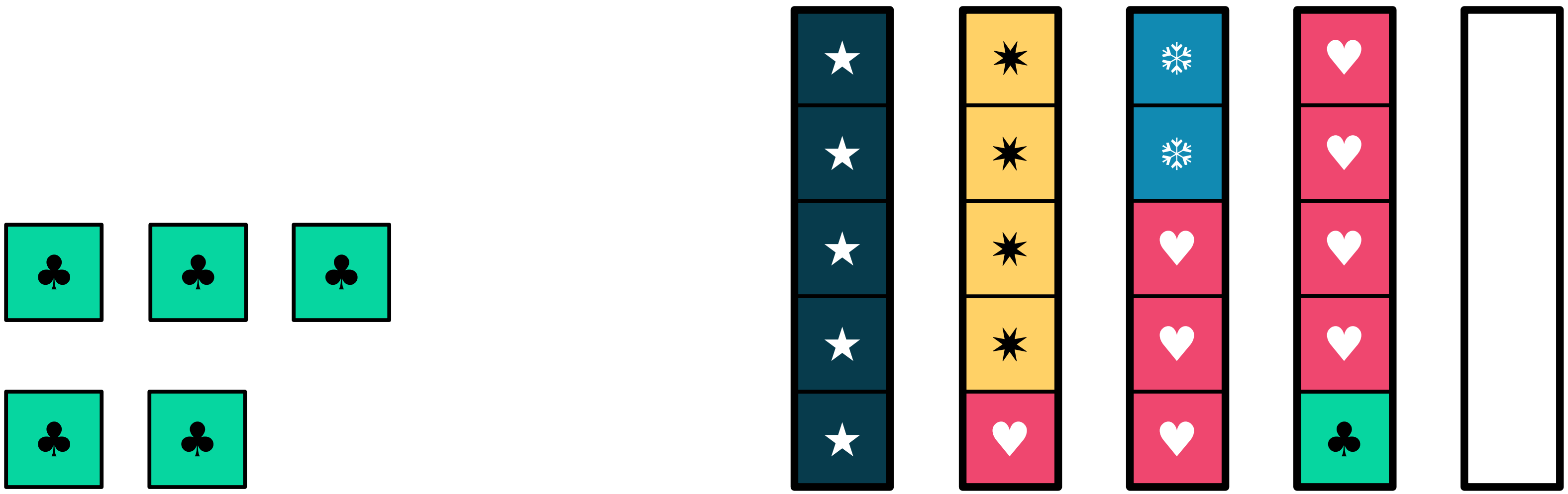

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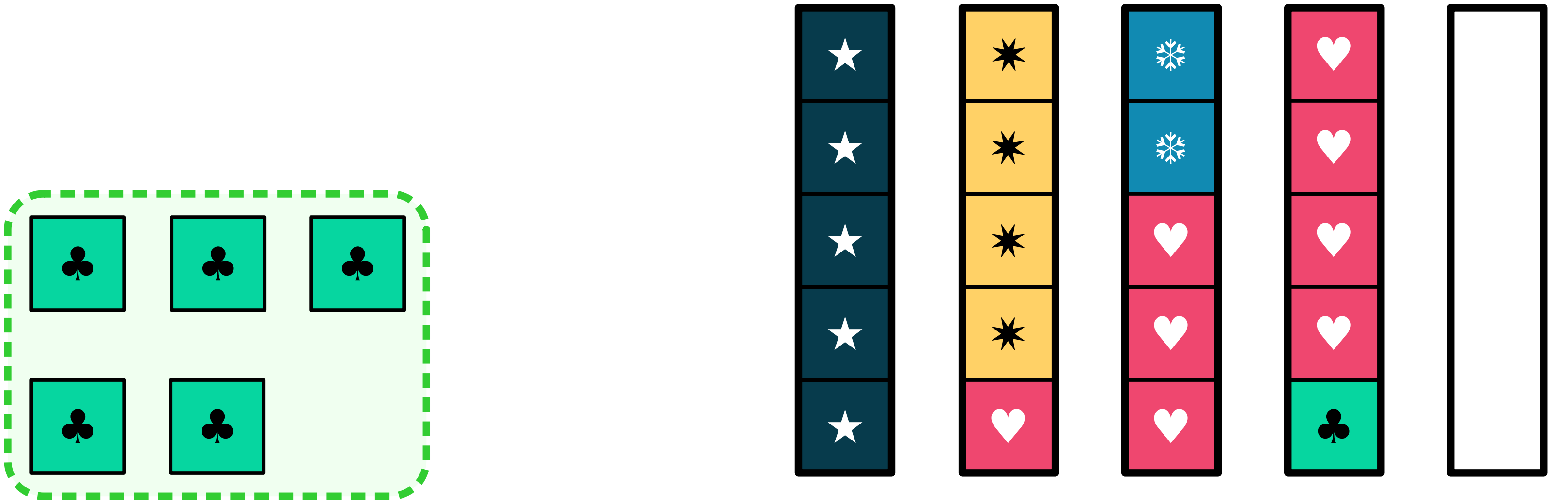

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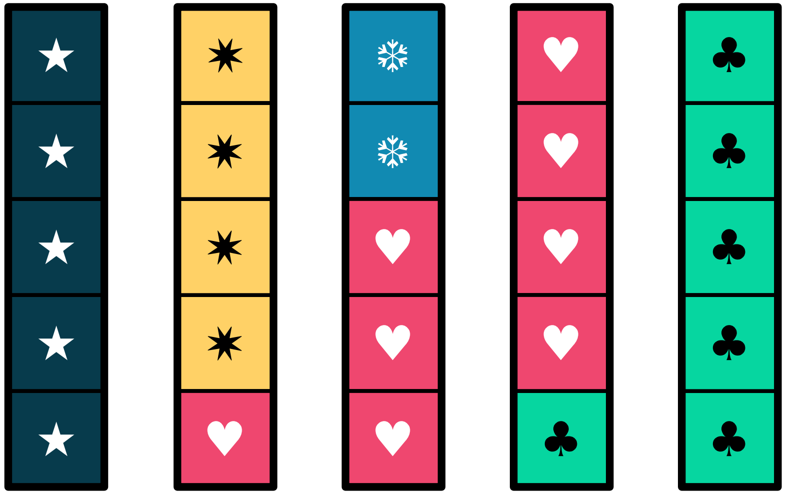
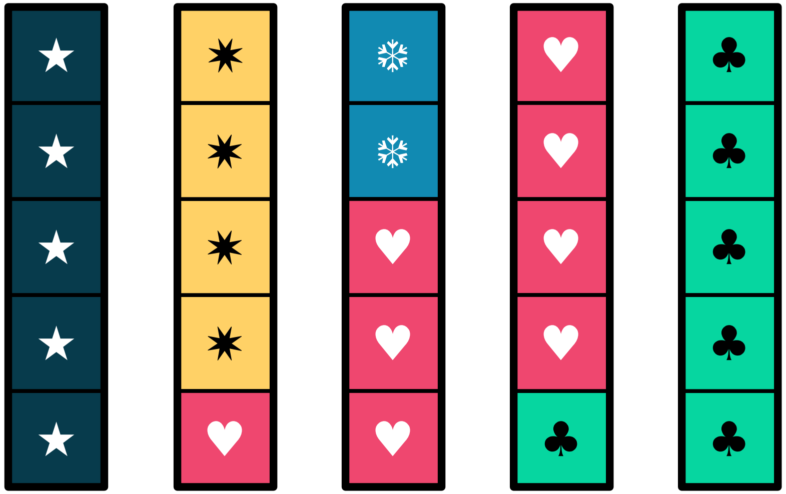


Idea: Begin with $5k+5$ potions and $k+1$ colors.
Find a way to remove five potions and one color.

Find a color that appears five or fewer times.

If it's exactly five times, place all potions of that color on a single shelf.

Otherwise, place all potions of that color and "top off" with potions of another color.



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Theorem: Every group of $5n$ potions of n colors has a good split.

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As our base case, we prove $P(0)$, that any group of 0 potions of 0 colors has a good split.

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> We need to find a color that appears five or fewer times. What mathematical tool guarantees such a color exists?

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Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions.

A nice abbreviation of "generalized pigeonhole principle."

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Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$.

Case 2: $p < 5$.

Theorem: Every group of $5n$ potions of n colors has a good split.

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Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$. Place all five purple potions onto their own shelf.

Case 2: $p < 5$.

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Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$. Place all five purple potions onto their own shelf.

Case 2: $p < 5$.

> We need to argue there's some other color that we can use to "top off" the shelf of the purple potions. How do we do that?

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Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$. Place all five purple potions onto their own shelf.

Case 2: $p < 5$. By the GPHP, there is some other color (call it **green**) with $g \geq 5$ potions.

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As our base case, we prove $P(0)$, that any group of 0 potions of 0 colors has a good split. Pick any group of 0 potions. Placing them onto 0 shelves satisfies the requirement of a good split, so $P(0)$ holds.

For our inductive step, pick some $k \in \mathbb{N}$ and assume $P(k)$ holds: any group of $5k$ potions of k colors has a good split. We will prove $P(k+1)$: that any group of $5k+5$ potions of $k+1$ colors has a good split.

Pick any group of $5k+5$ potions of $k+1$ colors. By the GPHP, there is a color (call it **purple**) with $p \leq 5$ potions. We consider two cases:

Case 1: $p = 5$. Place all five purple potions onto their own shelf.

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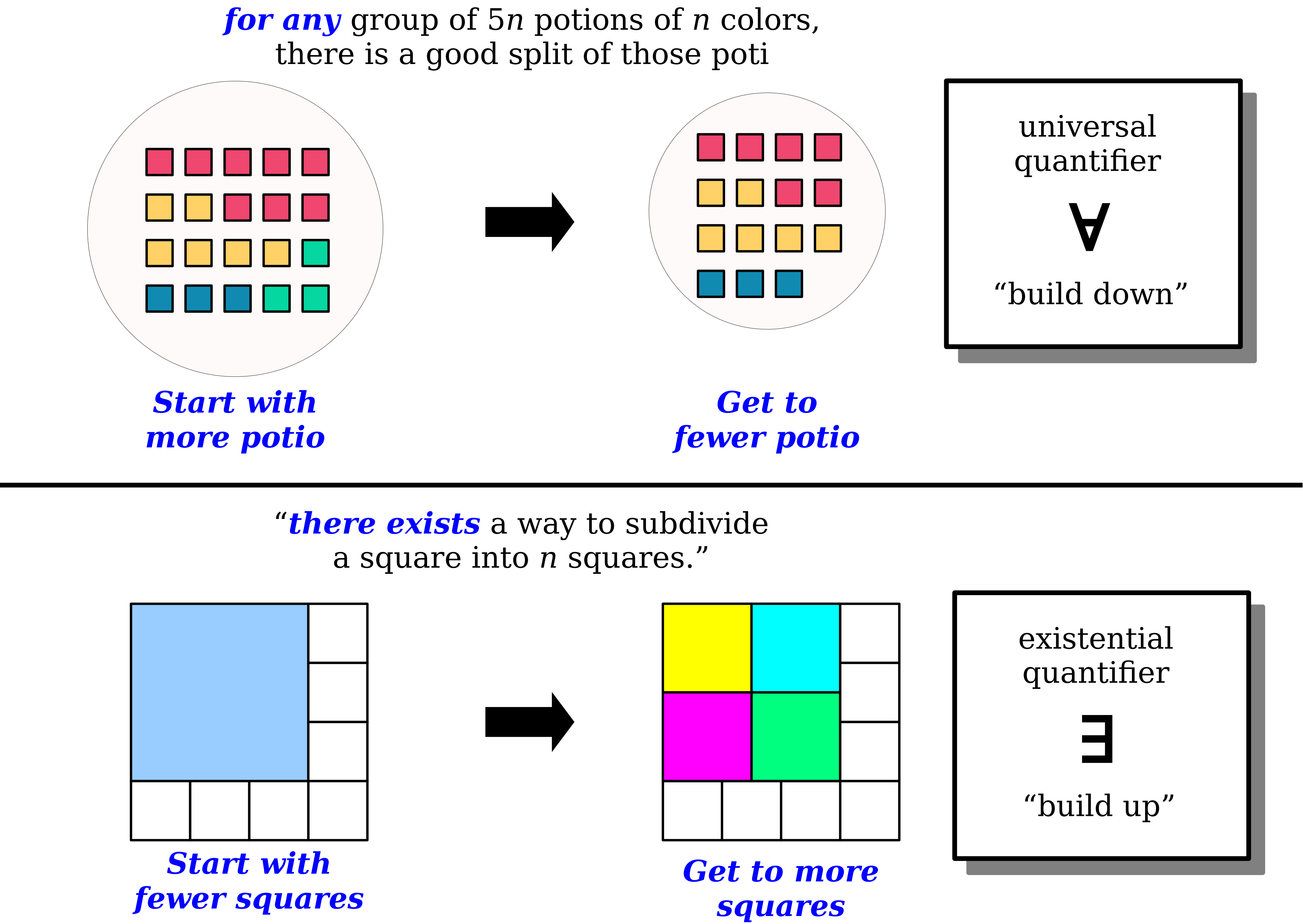
A Neat Application

- This result on colored potions forms the basis for the ***alias method***, a fast algorithm for simulated rolls of a loaded die in software.
- This in turn has applications throughout computer science.
- Want to learn more? Check out ***this blog post***, which shows how to apply this result.

An Observation



for any group of 5*n* potions of *n* colors,
there is a good split of those poti
universal
quantifier
∀
“build down”
Start with
more potio
Get to
fewer potio
“***there exists*** a way to subdivide
a square into *n* squares.”
existential
quantifier
∃
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Start with
fewer squares
Get to more
squares

Following the Rules

- When working with square subdivisions, our predicate looked like this:

  $P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with colored potions, our predicate looked like this:

  $P(n)$ is "***for any*** group of $5n$ potions of n colors, there is a good split of those potions."

- With squares, the quantifier is $\exists$. With potions, the first quantifier is $\forall$.
- This fundamentally changes the "feel" of induction.

Build Up with $\exists$

- In the case of squares, in our inductive step, we prove

  If

  there exists a subdivision into k squares,

  then

  there exists a subdivision into k+3 squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.
- Proving the consequent means finding some way to subdivide in to k+3 squares.
- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the colored potions case, in our inductive step, we prove

  If

  for all groups of $5k$ potions of k colors,

  there's a good split

  then

  for all groups of $5k+5$ potions of $k+1$ colors,

  there's a good split

- Assuming the antecedent means once we find $5k$ potions and k colors, we can group them into a good split.

Some Notes

- Not all predicates $P(n)$ will have the form outlined here.
  - That's okay! Just use the normal rules for assuming and proving things.
  - Think of these as quick shorthands rather than fundamentally new strategies.
- In all cases, assume $P(k)$ and prove $P(k+1)$.
  - All that changes is what you do to assume $P(k)$ and what you do to prove $P(k+1)$.
- When in doubt, consult the assume/prove table.
  - It really does work for all cases!

Quick check-in!

https://cs103.stanford.edu/pollev

Complete Induction

Guess what?

It’s time for

Mathematical

Calesthenics!

It’s time for

Mathematicalesthenics!

If you are the ***leftmost*** person
in your row, stand up right now.

Everyone else: stand up as soon as the
person to your left in your row stands up.

Thank you! Please be seated.

What Just Happened?

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(k) \to P(k+1)$.

Round Two!

If you are the ***leftmost*** person
in your row, stand up right now.

Everyone else: stand up as soon as
everyone left of you in your row stands up.

Thank you! Please be seated.

What Just Happened?

This is kinda like P(0).

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as ***everyone*** left of you in your row stands up.

What sort of sorcery is this?

Let P be some predicate. The ***principle of complete induction*** states that if

If it starts true…

$P(0)$ is true

…and it stays true…

and

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

…then it's always true.

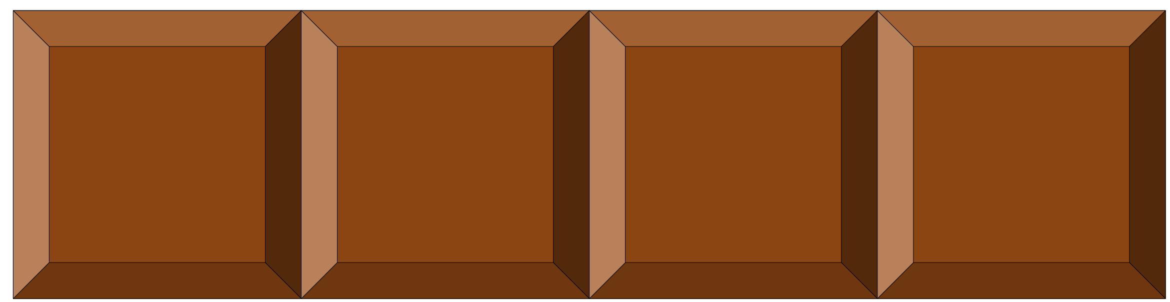
Mathematical Induction

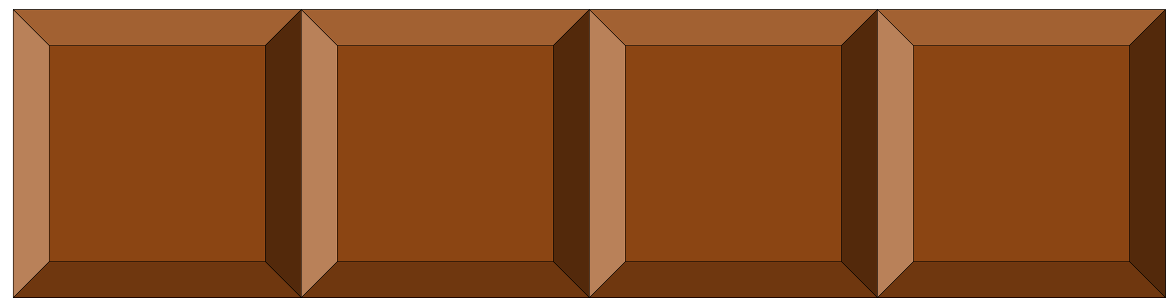
- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

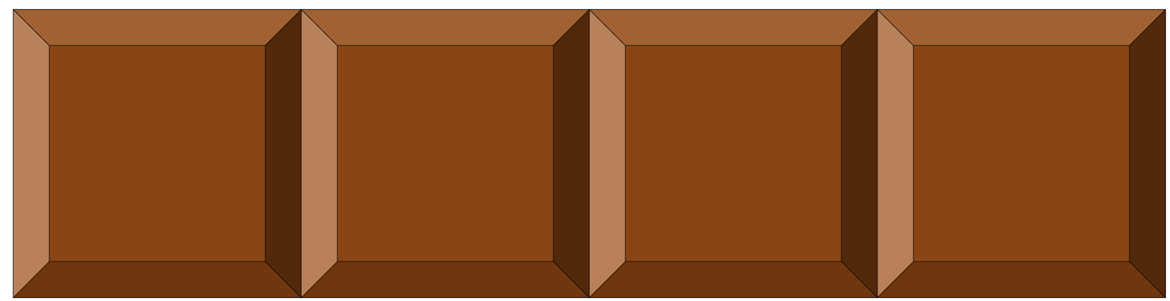
Complete Induction

- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that ***P*(0), *P*(1), *P*(2), …, and *P*(*k*)** are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

An Example: ***Eating a Chocolate Bar***



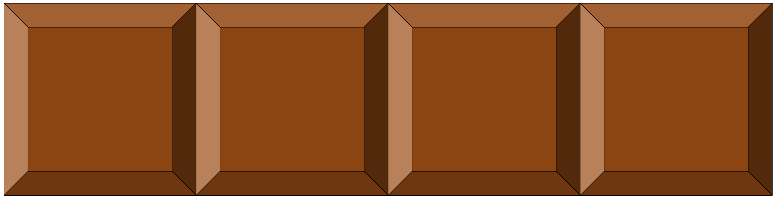


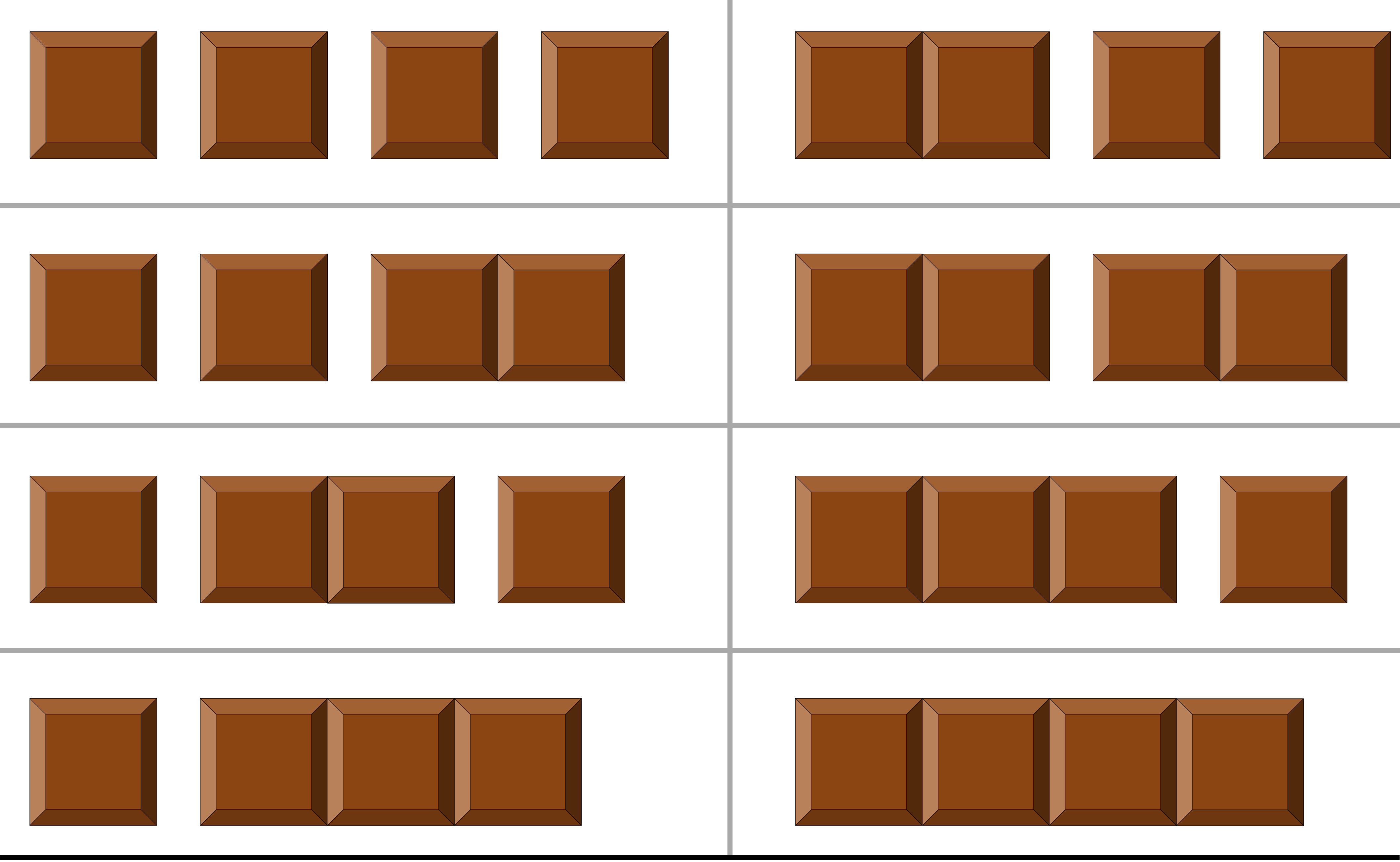


Eating a Chocolate Bar

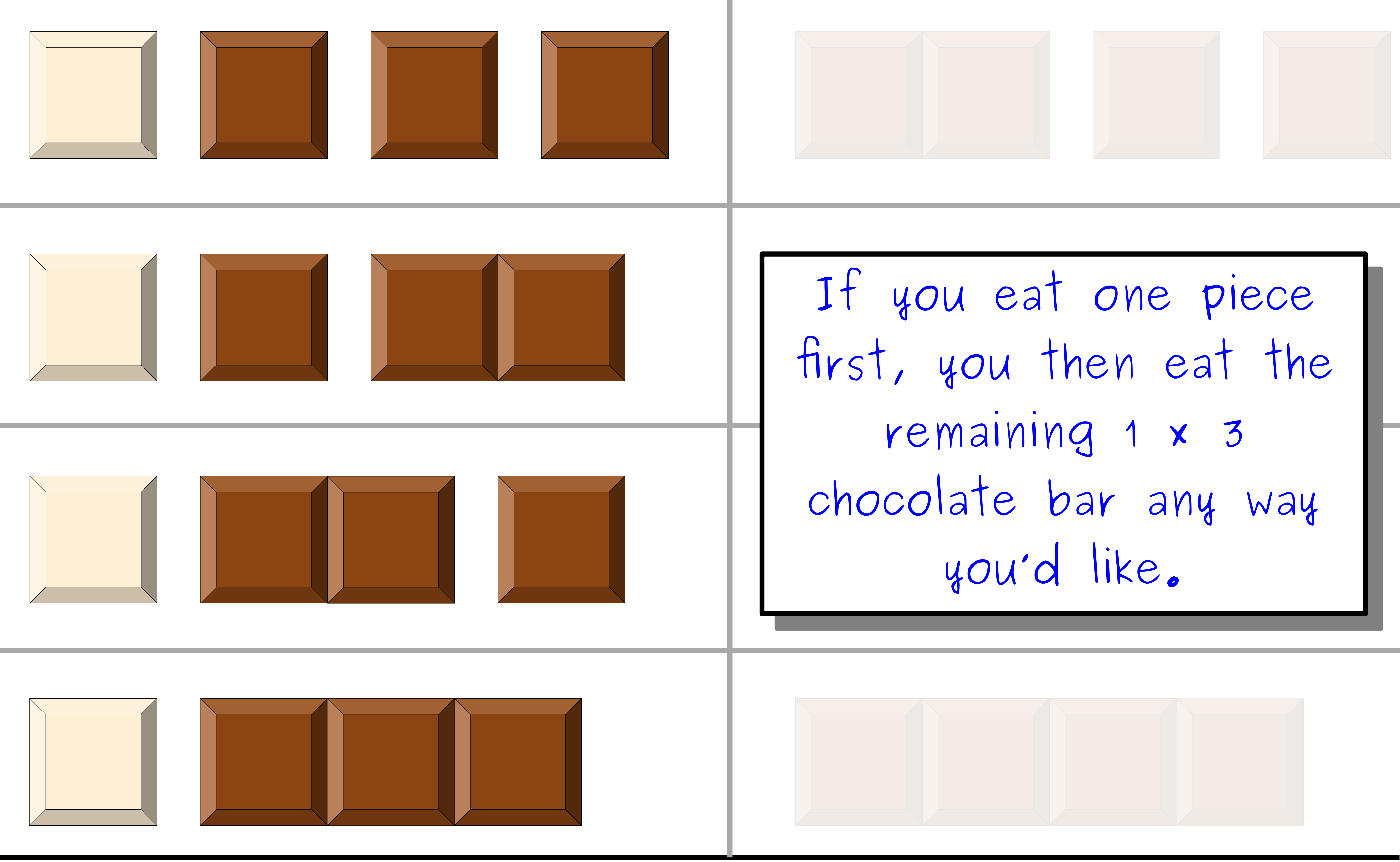
Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a...
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?





There are eight ways to eat a 1×4 chocolate bar.



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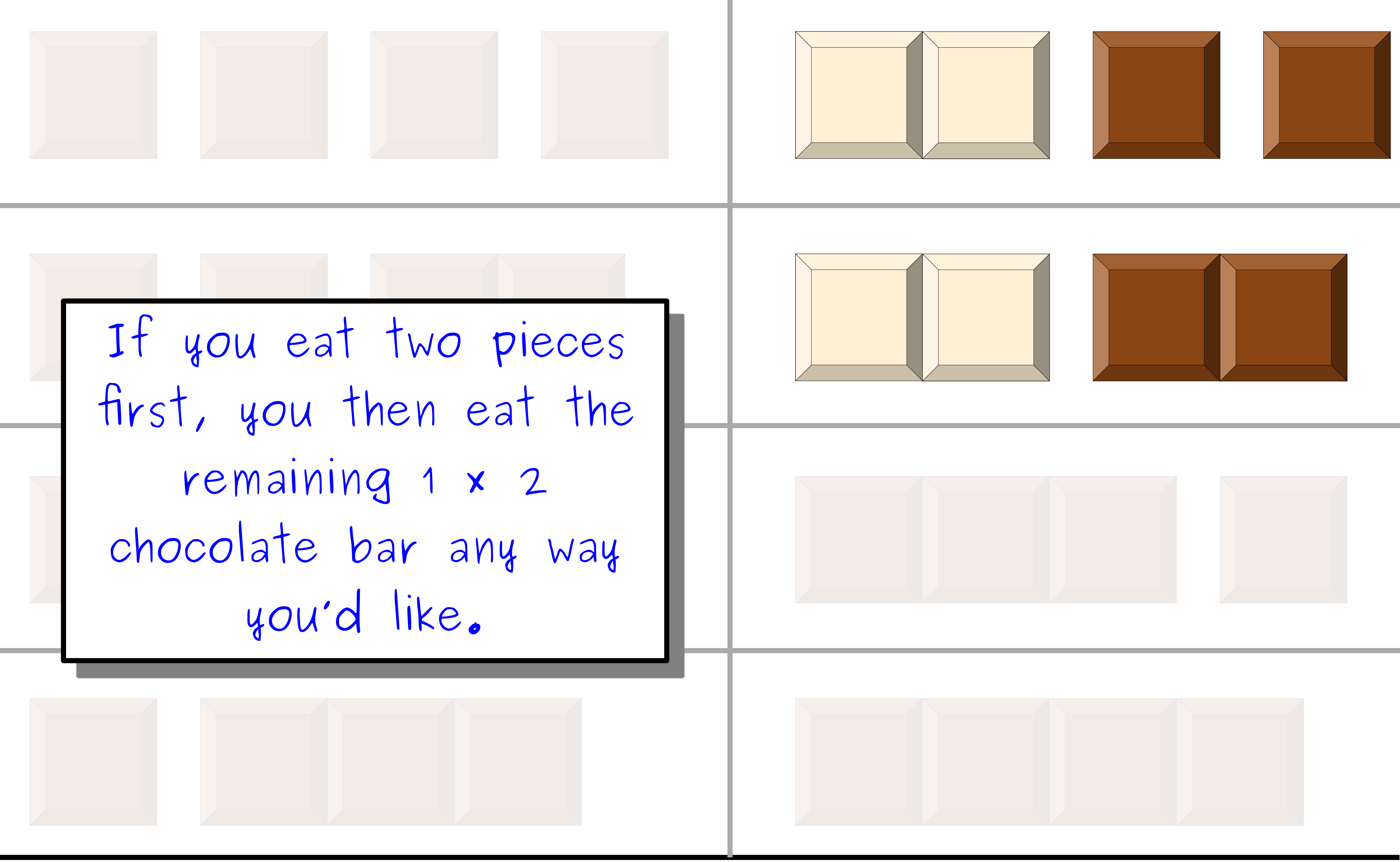

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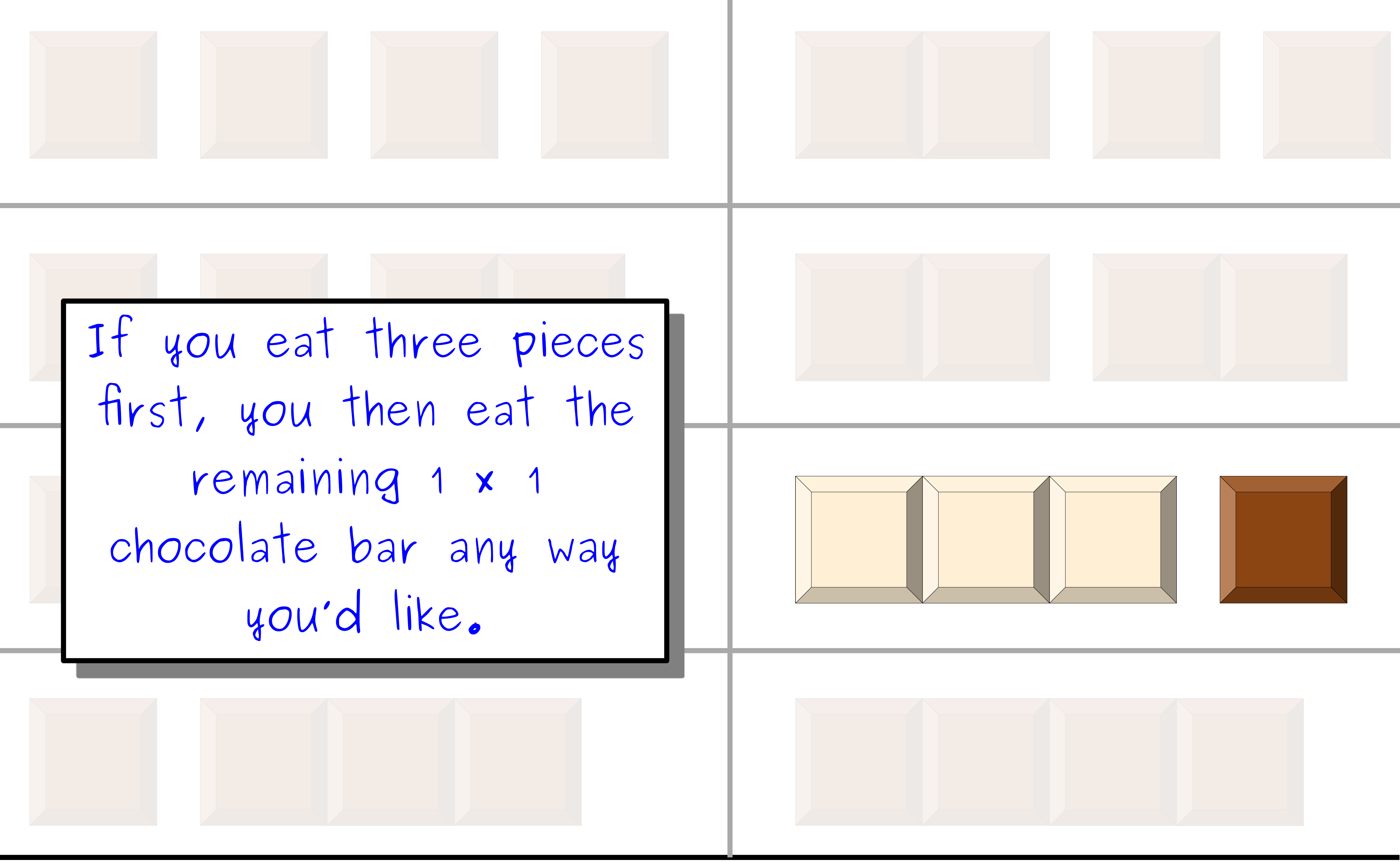

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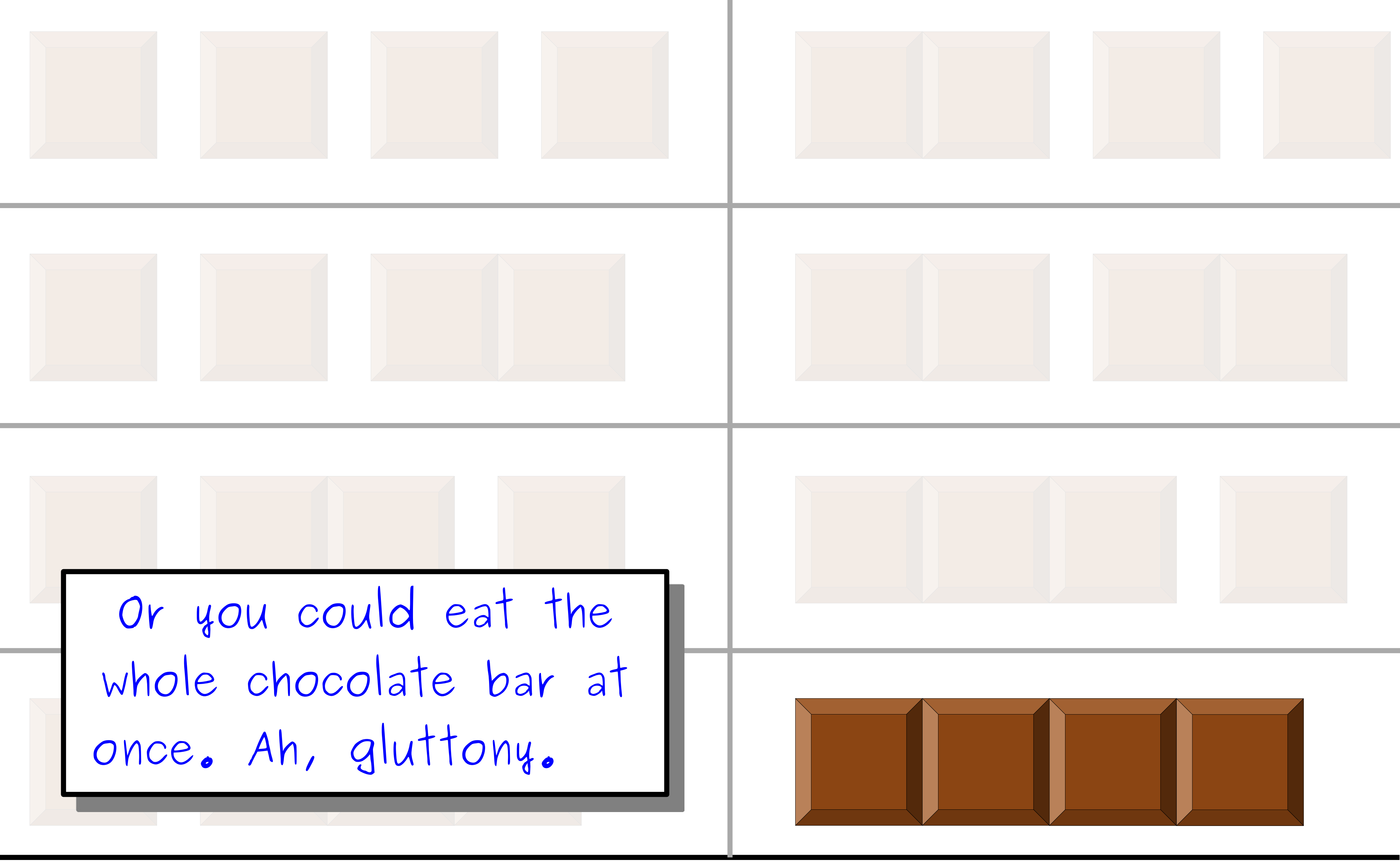

There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1×1 chocolate bar,
  - 2 ways to eat a 1×2 chocolate bar,
  - 4 ways to eat a 1×3 chocolate bar, and
  - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

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This is the spot where we need complete induction. We know we have a smaller chocolate bar, but its size isn't necessarily k.

Theorem: For any natural number $n \geq 1$, the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}.

Proof: Let $P(n)$ be "the number of ways to eat a $1 \times n$ chocolate bar from left to right is 2^{n-1}." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, f

As our base case … to eat a 1×1 chocolate bar fro… n here is to eat the entire chocol… o eat it, as needed.

To be able to apply our inductive hypothesis, we need to know that $k+1-r$ is between 1 and k.

For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, …, and $P(k)$ are true. We need to show $P(k+1)$ is true, that the number of ways to eat a $1 \times (k+1)$ chocolate bar is 2^k.

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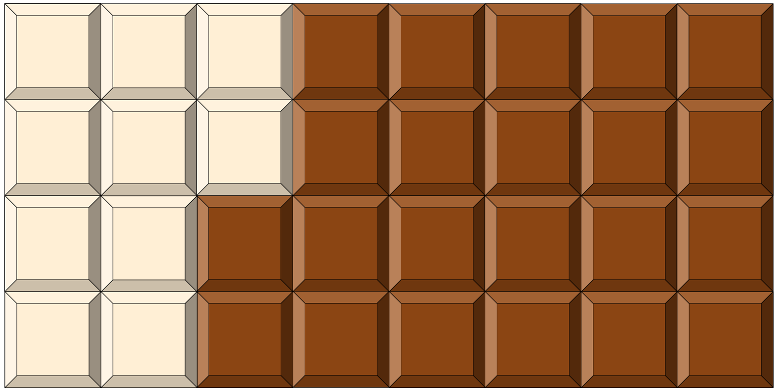
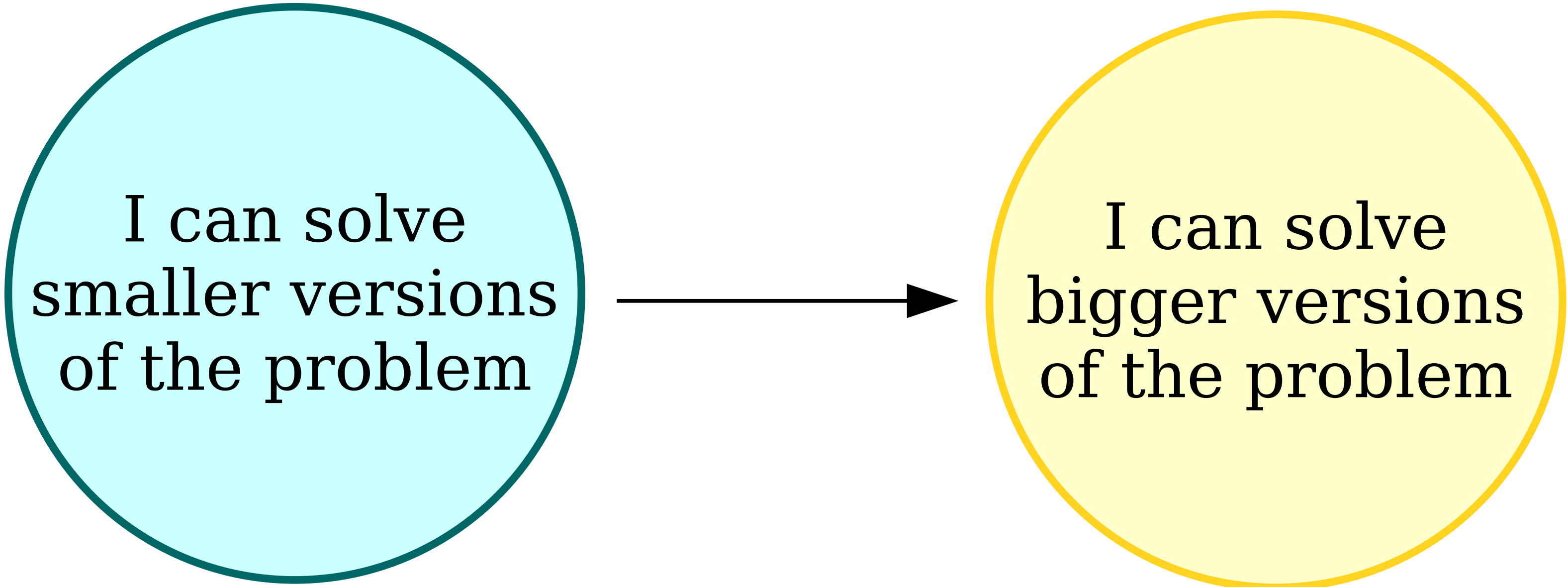
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?



- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

Induction vs. Complete Induction



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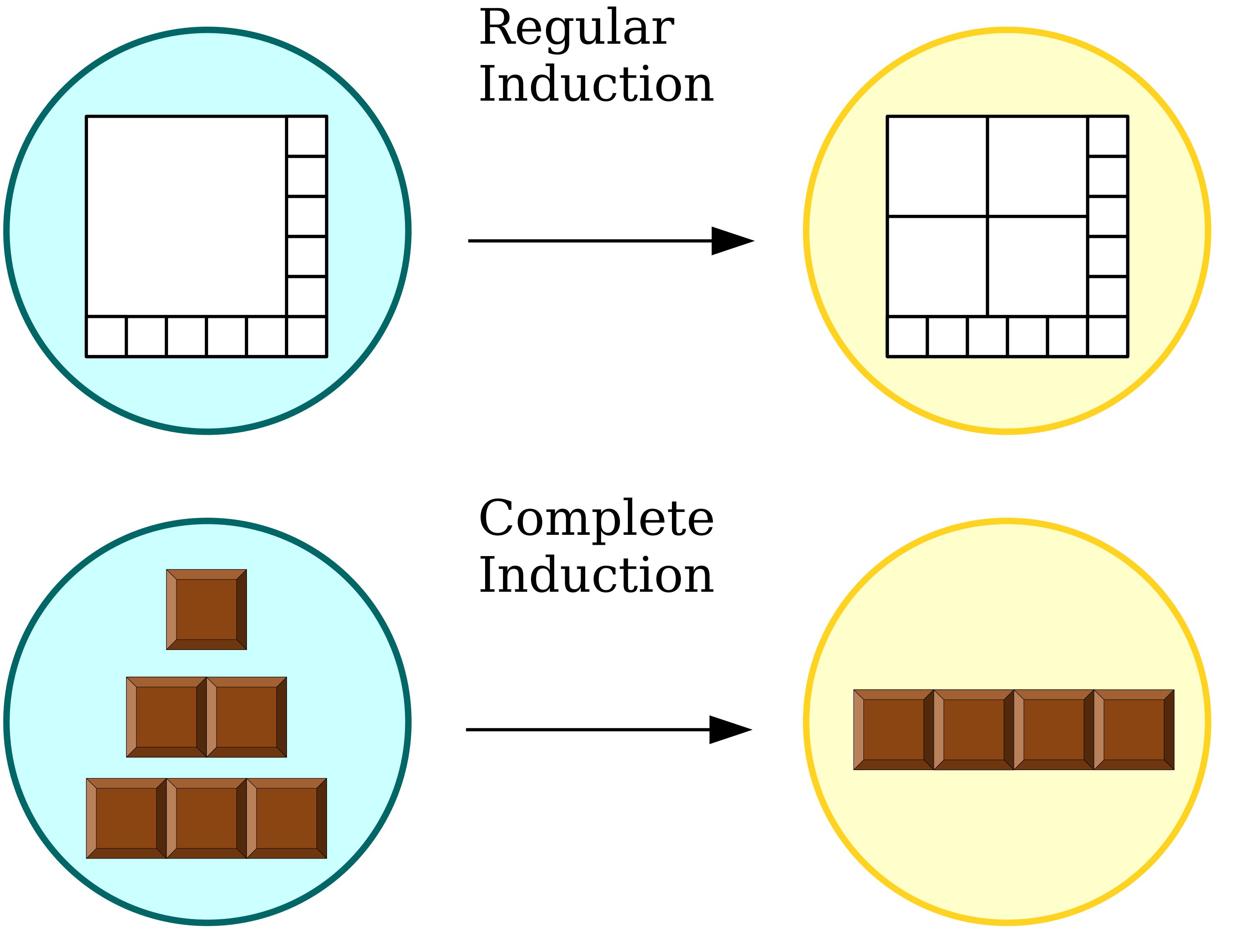
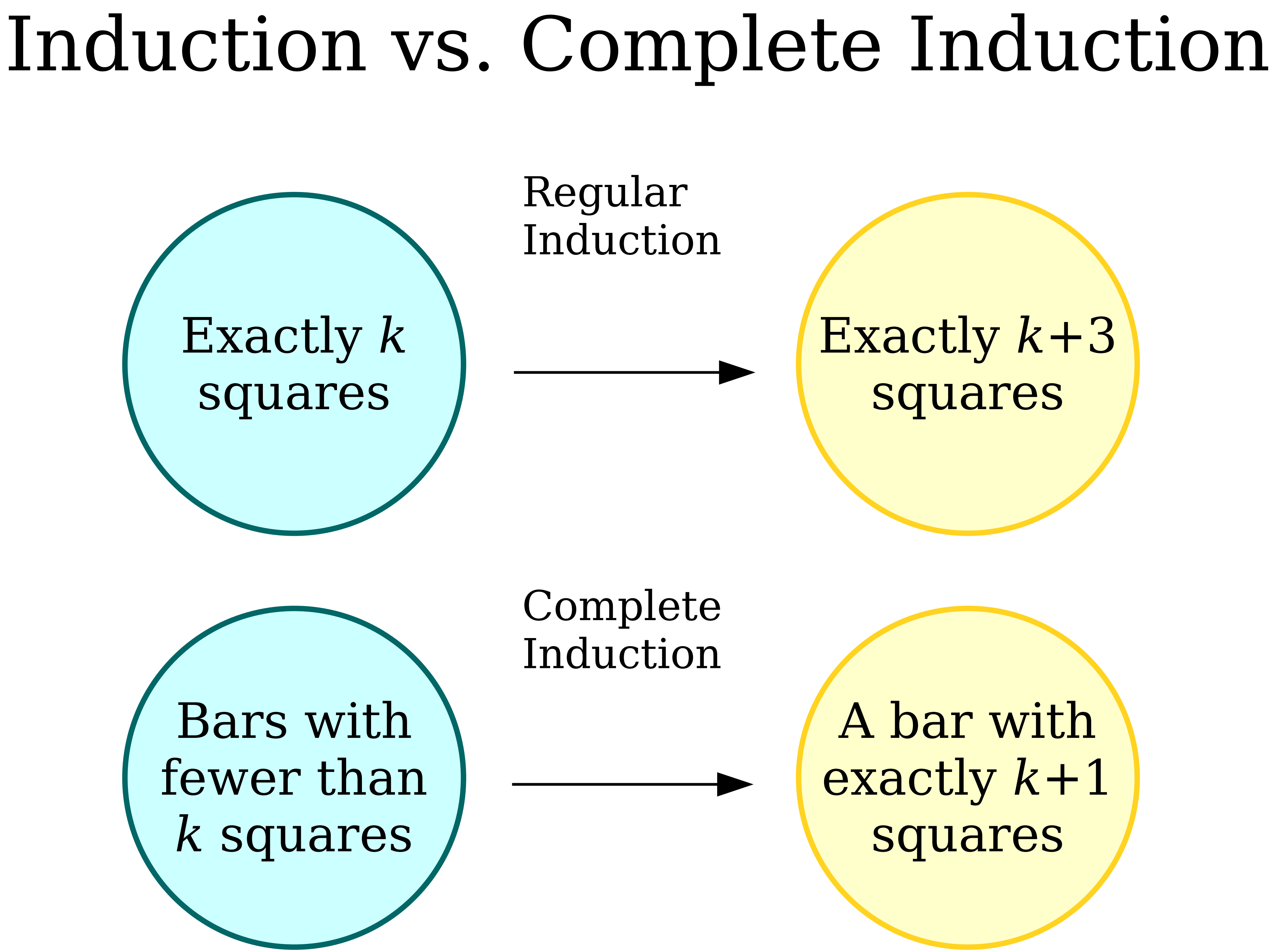
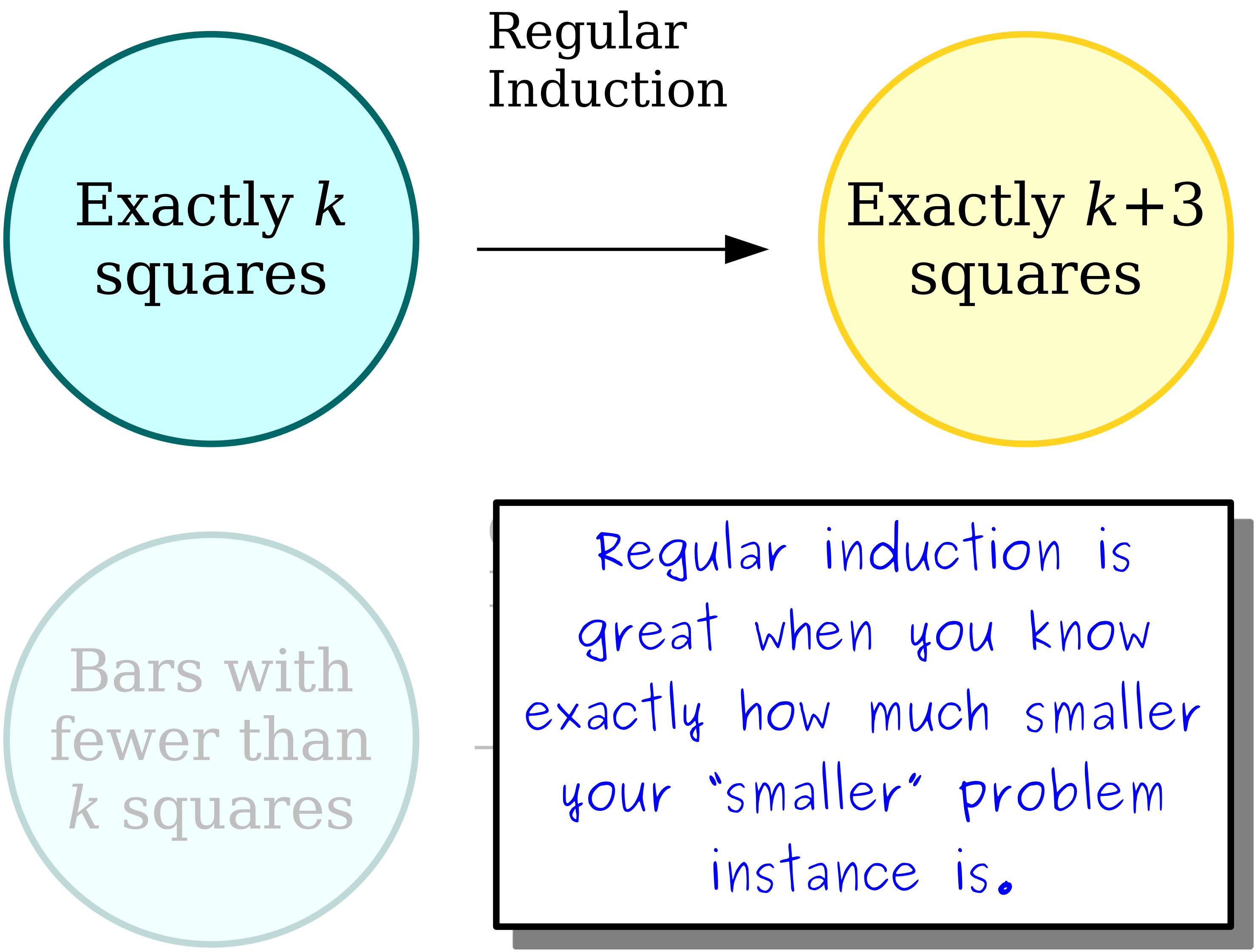


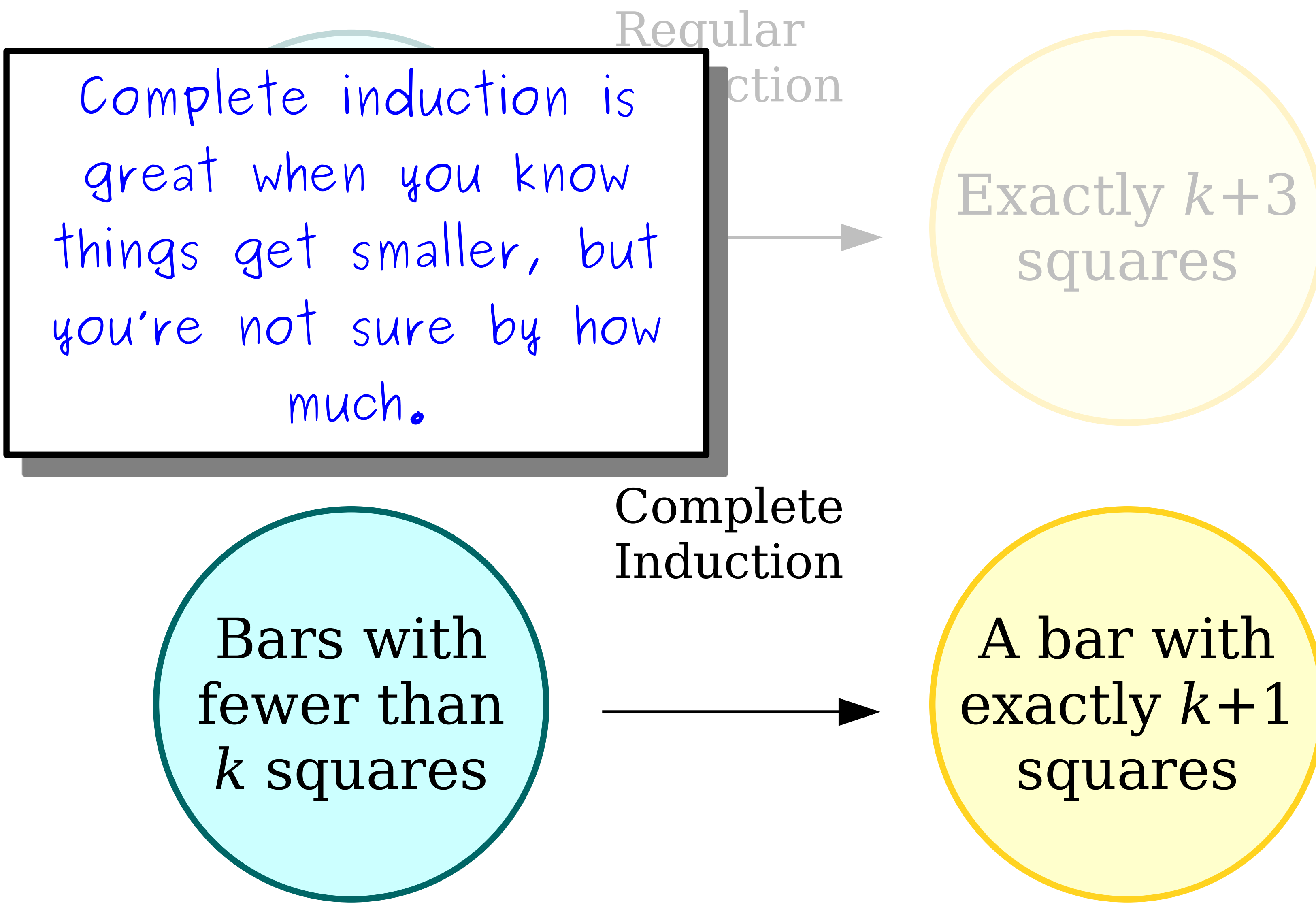

Induction vs. Complete Induction
Regular Induction
Exactly k squares
Exactly k+3 squares
Complete Induction
Bars with fewer than k squares
A bar with exactly k+1 squares

Induction vs. Complete Induction



Regular induction is great when you know exactly how much smaller your "smaller" problem instance is.

Induction vs. Complete Induction



An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.